On the r-th dispersionless Toda hierarchy I: Factorization problem, symmetries and some solutions

Manuel Mañas
Departamento de Física Teórica II, Universidad Complutense
28040-Madrid, Spain
email: manuel@darboux.fis.ucm.es

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Abstract

For a family of Poisson algebras, parametrized by $r \in \mathbb{Z}$, and an associated Lie algebraic splitting, we consider the factorization of given canonical transformations. In this context we rederive the recently found r-th dispersionless modified KP hierachies and r-th dispersionless Dym hierarchies, giving a new Miura map among them. We also found a new integrable hierarchy which we call the r-th dispersionless Toda hierarchy. Moreover, additional symmetries for these hierarchies are studied in detail and new symmetries depending on arbitrary functions are explicitly constructed for the r-th dispersionless KP, r-th dispersionless Dym and r-th dispersionless Toda equations. Some solutions are derived by examining the imposition of a time invariance to the potential r-th dispersionless Dym equation, for which a complete integral is presented and therefore an appropriate envelope leads to a general solution. Symmetries and Miura maps are applied to get new solutions and solutions of the r-th dispesionless modified KP equation.

1 Introduction

The study of dispersionless integrable hierarchies is a subject of increasing activity in the theory of integrable systems. This was originated in several sources, let us metion mention here the pioneering work of Kodama and Gibbons [14] on the dispersionless KP, of Kupershimdt on the dispersionless modified KP [18] and the role of Riemann invariants and hodograph transformations found by Tsarev [9]. The important work of Takasaki and Takabe, [22], [21] and [23] which gave the Lax formalism, additional symmetries, twistor formulation of the dispersionless KP and dispersionless Toda hierarchies. For the dispersionless Dym (or Harry Dym) equation see [24]. The appearance of dispersionless systems in topological field theories is also an important issue, see [17] and [4]. More recent progress appears in relation with the theory of conformal maps [10] and [25], quasiconformal maps and $\bar{\partial}$- formulation [15], reductions of several type [19] and [12], additional symmetries [20] and twistor equations [13], on hodograph equations for the Boyer–Finley equation [7] and its applications in General Relativity, see also [5] and [6]. It is

also remarkable the approach given in [8] to the theory. Finally, we comment the contribution on electrodynamics and the dispersionless Veselov–Novikov equation [16].

Recently, a new Poisson bracket and associated Lie algebra splitting, therefore using a Lax formalism and an r-matrix approach, was presented in [1] to construct new dispersionless integrable hierarchies and latter on, see [2], the theory was further extended. We must remark that since the work [11] and [24] it was known the possibility for a r-matrix formulation of dispersionless integrable systems.

In this paper we shall use this new splitting together with a standard technique in the theory of integrable hierarchies, the factorization problem, to get a new dispersionless integrable hierarchy, which we call the r-th dispersionless Toda hierarchy. For that aim we consider the Lie group of canonical transformations associated with a particular Poisson bracket together with a factorization problem induced by a r-matrix associated with a canonical splitting of the corresponding algebra of symplectic vector fields [1]. These new hierarchies contains dispersionless integrable equations derived previously in [1] —which we call r-th dispersionless modified KP and r-th dispersionless Dym equations (for $r = 0$ we get the well-known dispersionless modified KP and dispersionless Dym equations)—. However, we found new integrable hierarchies, which we decided to name as r-th dispersionless Toda hierarchy (as for $r = 1$ we get the well-known dispersionless Toda hierarchy).

In [3] the Miura map among the dispersionless modified KP and dispersionless Dym equations was presented, here we extend those results to the present context. We also study in detail the additional symmetries of these hierarchies and in particular for the three integrable equations: r-th dispersionless modified KP, r-th dispersionless Dym and r-th dispersionless Toda equations we get new explicit symmetries depending on arbitrary functions, in the spirit of the symmetries given in [5] for the dispersionless KP equation.

Latter we find a complete integral for the t_2-reduction of the potential r-th dispersionless Dym equation. Thus, using the method of the complete solution, an appropriate envelope leads to its general solution. Then, when the symmetries are applied we get more general solutions, non-t_2 invariant of the potential r-th dispersionless Dym equation. Using the Miura map new solutions of the r-th dispersionless modified KP equation are gotten and the corresponding functional symmetries are applied to get more general families of solutions. Finally, we also derive from the factorization problem twistor equations for these integrable hierarchies.

The layout of this paper is as follows. In §2 we consider the Lie algebra setting, the factorization problem and its differential description, Lax functions and Zakharov–Shabat formulation are given, as well. Then, in §3 the corresponding integrable hierarchies are derived together with the Miura map. Next, in §4, we introduce Orlov functions and show how the factorization problem constitutes a simple framework to derive the corresponding additional symmetries of the integrable hierarchies. In particular, additional symmetries (which depend on arbitrary functions of t_2) of the integrable dispersionless equations discussed in §3 are found. Finally, §5 is devoted to some solutions of these integrable hierarchies and §6 to a twistor formulation of these dispersionless integrable hierarchies derived from the factorization problem. In a forthcoming paper we will give a twistor formulation of these hierarchies which is derived independently of the factorization problem.

2 Factorization problem and its differential versions

We shall work with the Lie algebra $\mathfrak{g}$ of Laurent series $H(p,x) := \sum_{n\in\mathbb{Z}} u_n(x)p^n$ in the variable $p \in \mathbb{R}$ with coefficients depending on the variable $x \in \mathbb{R}$, with Lie commutator given by the following Poisson bracket [1]

$$\{H_1, H_2\} = p^r\Big(\frac{\partial H_1}{\partial p}\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial p}\Big), \quad r \in \mathbb{Z}.$$

Observe that for each $r \in \mathbb{Z}$ we are dealing with a different Lie algebra; notice also that this Poisson bracket is associated with the the following sympletic form

$$\omega := p^{-r}\mathrm{d}p \wedge \mathrm{d}x.$$

2.1 The Lie algebra splitting

We shall use the following triangular type splitting of $\mathfrak{g}$ into Lie subalgebras

$$\mathfrak{g} = \mathfrak{g}_> \oplus \mathfrak{g}_{1-r} \oplus \mathfrak{g}_< \tag{1}$$

where

$$\mathfrak{g}_{\gtrless} := \mathbb{C}\{u_n(x)p^n\}_{n\gtrless(1-r)}, \quad \mathfrak{g}_{1-r} := \mathbb{C}\{u(x)p^{1-r}\},$$

and therefore fulfil the following property

$$\{\mathfrak{g}_{\gtrless}, \mathfrak{g}_{1-r}\} = \mathfrak{g}_{\gtrless}.$$

If we define the Lie subalgebra $\mathfrak{g}_{\geqslant}$ as

$$\mathfrak{g}_{\geqslant} := \mathfrak{g}_{1-r} \oplus \mathfrak{g}_>$$

we have the direct sum decomposition of the Lie algebra $\mathfrak{g}$ given by

$$\mathfrak{g} = \mathfrak{g}_< \oplus \mathfrak{g}_{\geqslant}. \tag{2}$$

We remark that allowing the variable p to take values in $\mathbb{C}$ we have the following interpretation for the above splitting (2). Suppose that $\mathfrak{g}$ is the set of analytic functions in some annulus of $p = 0$. Then, $H \in \mathfrak{g}_<$ iff $H(p)p^{-1+r}$ is an analytic function outside the annulus which vanish at $p = \infty$. On the opposite a function $H \in \mathfrak{g}_{\geqslant}$ if $H(p)p^{-1+r}$ has an analytic extension inside the annulus.

Observe that the induced Lie commutator in $\mathfrak{g}_{1-r}$ is

$$\{f(x)p^{1-r}, g(x)p^{1-r}\} = (1-r)W(f,g),$$

where $W(f,g) := fg_x - gf_x$ is the Wrońskian of f and g. Only when $r = 1$ the Lie subalgebra $\mathfrak{g}_{1-r}$ is an Abelian Lie subalgebra.

An alternative realization of $\mathfrak{g}$ is through the adjoint action

$$\begin{aligned}\mathrm{ad}: \quad \mathfrak{g} &\to \mathfrak{X}(\mathbb{R}^2),\\ H &\mapsto \mathrm{ad}_H := \{H, \cdot\}.\end{aligned}$$

so that

$$\mathrm{ad}_H = p^r\frac{\partial H}{\partial p}\frac{\partial}{\partial x} - p^r\frac{\partial H}{\partial x}\frac{\partial}{\partial p}.$$

2.2 The factorization problem

Given an element $H \in \mathfrak{g}$ the corresponding vector field ad_H generates a symplectic diffeormorphism (canonical transformation) Φ_H, given by

$$\Phi_H := \sum_{n=0}^{\infty} \frac{1}{n!}(\mathrm{ad}_H)^n.$$

This transformation corresponds to the element $h = \exp(H)$ belonging to the local Lie group G generated by $\mathfrak{g}$. In fact, $\Phi_H = \mathrm{Ad}_h$, the adjoint action of the Lie group G on $\mathfrak{g}$.

In what follows we shall denote by $G_<, G_{1-r}, G_>$ and $G_\geqslant$ the local Lie groups corresponding to the Lie algebras $\mathfrak{g}_<, \mathfrak{g}_{1-r}, \mathfrak{g}_>$ and $\mathfrak{g}_\geqslant$, respectively.

Given $h, \bar{h} \in G$ in the local Lie group G the finding of $h_< \in G_<$ and $h_\geqslant \in G_\geqslant$ such that the following factorization holds

$$h_< \cdot h = h_\geqslant \cdot \bar{h}, \tag{3}$$

will play a pivotal role in what follows.

Furthermore, given two sets of deformation parameters $(t_1, t_2, \dots)$ and $(\bar{t}_1, \bar{t}_2, \dots))$ and corresponding elements in the Lie algebra $\mathfrak{g}$

$$t(p) := t_1 p^{2-r} + t_2 p^{3-r} + \cdots \in \mathfrak{g}_>, \quad \bar{t}(p) := \bar{t}_1 p^{-r} + \bar{t}_2 p^{-r-1} + \cdots \in \mathfrak{g}_<,$$

we shall analyze the following deformation of (3):

$$\psi_< \cdot \exp(t) \cdot h = \psi_\geqslant \cdot \exp(\bar{t}) \cdot \bar{h}. \tag{4}$$

Notice that there is no loss of generality if we set $\bar{h} = 1$ in (4) so that

$$\exp(t) \cdot h \cdot \exp(-\bar{t}) = \psi_<^{-1} \cdot \psi_\geqslant. \tag{5}$$

2.3 Differential consequences of the factorization problem

A possible way to study (4) is by analyzing its differential versions; i. e., by taking right derivatives. Given a derivation ∂ of a Lie algebra $\mathfrak{g}$ one defines the corresponding right-derivative in the associated local Lie group by

$$\partial h \cdot h^{-1} := \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(\mathrm{ad}\, H)^n(\partial H), \quad h := \exp(H), \quad H \in \mathfrak{g}. \tag{6}$$

Hence, by taking right-derivatives of (4) with respect to

$$\partial_n := \frac{\partial}{\partial t_n}, \quad \bar{\partial}_n := \frac{\partial}{\partial \bar{t}_n},$$

we get

$$\partial_n \psi_< \cdot \psi_<^{-1} + \mathrm{Ad}_{\psi_<}(p^{n+1-r}) = \partial_n \psi_\geqslant \cdot \psi_\geqslant^{-1}, \tag{7}$$
$$\bar{\partial}_n \psi_< \cdot \psi_<^{-1} = \bar{\partial}_n \psi_\geqslant \cdot \psi_\geqslant^{-1} + \mathrm{Ad}_{\psi_\geqslant}(p^{1-r-n}). \tag{8}$$

If we further factorize

$$\psi_{\geqslant} = \psi_{1-r} \cdot \psi_{>}$$

equation (7) decomposes –according with (1)– in the following three equations

$$\partial_n \psi_{<} \cdot \psi_{<}^{-1} + P_{<} \operatorname{Ad}_{\psi_{<}} p^{n+1-r} = 0, \tag{9}$$

$$P_{1-r} \operatorname{Ad}_{\psi_{<}} p^{n+1-r} = \partial_n \psi_{1-r} \cdot \psi_{1-r}^{-1}, \tag{10}$$

$$P_{>} \operatorname{Ad}_{\psi_{<}} p^{n+1-r} = \operatorname{Ad}_{\psi_{1-r}} \big(\partial_n \psi_{>} \cdot \psi_{>}^{-1}\big). \tag{11}$$

Similar considerations applied to (8) lead to

$$\bar\partial_n \psi_{<} \cdot \psi_{<}^{-1} = \operatorname{Ad}_{\psi_{1-r}} P_{<} \operatorname{Ad}_{\psi_{>}} p^{1-r-n}, \tag{12}$$

$$0 = \bar\partial_n \psi_{1-r} \cdot \psi_{1-r}^{-1} + \operatorname{Ad}_{\psi_{1-r}} P_{1-r} \operatorname{Ad}_{\psi_{>}} p^{1-r-n}, \tag{13}$$

$$0 = \bar\partial_n \psi_{>} \cdot \psi_{>}^{-1} + P_{>} \operatorname{Ad}_{\psi_{>}} p^{1-r-n}. \tag{14}$$

Now we shall show that we can interchange the roles of $\mathfrak{g}_{<}$ and $\mathfrak{g}_{\geqslant}$. For this aim we introduce the map

$$\begin{aligned} F : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (p, x) &\mapsto (p' := 1/p, x' = -x) \end{aligned} \tag{15}$$

that together with $r' = 2 - r$ may be considered as a "canonical" transformation in the sense that the canonical form $\Omega = p^{-r} \mathrm{d}p \wedge \mathrm{d}x$ transforms onto

$$\Omega' = (p')^{-r'} \mathrm{d}p' \wedge \mathrm{d}x'.$$

If we denote

$$\mathfrak{g}'_{\geqslant} := \mathbb{C}\{u_n(x)(p')^n\}_{n \geqslant (1-r')}, \quad \mathfrak{g}'_{1-r'} := \mathbb{C}\{u(x) p^{1-r'}\},$$

then

$$\mathfrak{g}_{>} \to \mathfrak{g}'_{<}, \quad \mathfrak{g}_{1-r} \to \mathfrak{g}'_{1-r'}, \quad \mathfrak{g}_{<} \to \mathfrak{g}'_{>}.$$

Observe that

$$\begin{aligned} t(p) &= t_1 p^{2-r} + t_2 p^{3-r} + \cdots \to t_1\, (p')^{-r'} + t_2\, (p')^{-r'-1} + \cdots = \bar t'(p'), \\ \bar t(p) &= \bar t_1 p^{-r} + \bar t_2 p^{-r-1} + \cdots \to \bar t_1\, (p')^{2-r'} + \bar t_2\, (p')^{3-r'} + \cdots = t'(p'), \end{aligned}$$

Thus, $t'_n = \bar t_n$ and $\bar t'_n = t_n$. Finally, the factorization (4) transforms as

$$\psi_{<} \cdot \exp(t) \cdot h = \psi_{1-r} \cdot \psi_{>} \cdot \exp(\bar t) \cdot \bar h \to \psi'^{-1}_{1-r'} \cdot \psi'_{>} \cdot \exp(\bar t') \cdot h = \psi'_{<} \cdot \exp(\bar t) \cdot \bar h \tag{16}$$

With these observations is easy to see that our map transforms equations (9), (10) and (11) into (14), (13) and (12), respectively.

2.4 Lax formalism and Zakharov–Shabat representation

Equations (9)-(14) can be given a Lax form, for that aim we first introduce the following Lax functions

$$\begin{aligned} L &:= \operatorname{Ad}_{\psi_{<}} p, \\ \bar\ell &:= \operatorname{Ad}_{\psi_{>}} p, \\ \bar L &:= \operatorname{Ad}_{\psi_{1-r}} \bar\ell = \operatorname{Ad}_{\psi_{\geqslant}} p \end{aligned} \tag{17}$$

in terms of which equations (7) and (8) read as

$$\begin{aligned}\partial_n\psi_< \cdot \psi_<^{-1} &= \partial_n\psi_\geqslant \cdot \psi_\geqslant^{-1} - L^{n+1-r},\\ \bar\partial_n\psi_\geqslant \cdot \psi_\geqslant^{-1} &= \bar\partial_n\psi_< \cdot \psi_<^{-1} - \bar L^{1-r-n}.\end{aligned}$$

So that

$$\begin{aligned}&\partial_n\psi_< \cdot \psi_<^{-1} = -P_< L^{n+1-r}, && \partial_n\psi_\geqslant \cdot \psi_\geqslant^{-1} = P_\geqslant L^{n+1-r},\\ &\bar\partial_n\psi_< \cdot \psi_<^{-1} = P_< \bar L^{1-r-n}, && \bar\partial_n\psi_\geqslant \cdot \psi_\geqslant^{-1} = -P_\geqslant \bar L^{1-r-n},\end{aligned} \tag{18}$$

and therefore the following Lax equations hold

$$\begin{aligned}&\partial_n L = \{-P_< L^{n+1-r}, L\}, && \partial_n \bar L = \{P_\geqslant L^{n+1-r}, \bar L\},\\ &\bar\partial_n L = \{P_< \bar L^{1-r-n}, L\}, && \bar\partial_n \bar L = \{-P_\geqslant \bar L^{1-r-n}, \bar L\}.\end{aligned} \tag{19}$$

To deduce these equations just recall that if $B = \operatorname{Ad}_\phi b$, and ∂ is a Lie algebra derivation, then $\partial B = \{\partial\phi \cdot \phi^{-1}, B\} + \operatorname{Ad}_\phi \partial b$.

For a further analysis (18) is essential to get the powers of L and $\bar L$. In the following proposition we shall show how the powers of the Lax functions are connected with $\Psi_<$, $\Psi_>$ and ξ, the infinitesimal generators of $\psi_<$, $\psi_>$ and ψ_{1-r}, respectively

$$\begin{aligned}\psi_< &:= \exp(\Psi_<), \quad \Psi_< := \Psi_1 p^{-r} + \Psi_2 p^{-r-1} + \cdots,\\ \psi_> &:= \exp(\Psi_>), \quad \Psi_> := \bar\Psi_1 p^{2-r} + \bar\Psi_2 p^{3-r} + \cdots,\\ \psi_{1-r} &:= \exp(\xi p^{1-r}).\end{aligned} \tag{20}$$

Proposition 1. *We can parameterize L^m, $\bar\ell^m$ and $\bar L^m$ in terms of Ψ_n, $\bar\Psi_n$ and ξ as follows*

$$L^m = p^m + u_{m,0}p^{m-1} + u_{m,1}p^{m-2} + u_{m,2}p^{m-3} + O(p^{m-4}), \quad p\to\infty \tag{21}$$

$$\bar\ell^m = p^m + \bar v_{m,0}p^{m+1} + \bar v_{m,1}p^{m+2} + \bar v_{m,2}p^{m+3} + O(p^{m+4}), \quad p\to 0, \tag{22}$$

$$\bar L^m = \bar u_{m,-1}p^m + \bar u_{m,0}p^{m+1} + \bar u_{m,1}p^{m+2} + \bar u_{m,2}p^{m+3} + O(p^{m+4}), \quad p\to 0 \tag{23}$$

where the first coefficients are

$$\begin{aligned}u_{m,0} =& -m\Psi_{1,x},\\ u_{m,1} =& m\Big(-\Psi_{2,x} + \frac{1}{2}\big(r\Psi_1\Psi_{1,xx} + (m-1)\Psi_{1,x}^2\big)\Big),\\ u_{m,2} =& m\Big(-\Psi_{3,x} + \frac{1}{2}\big((r+1)\Psi_2\Psi_{1,xx} + (2m-3)\Psi_{1,x}\Psi_{2,x} + r\Psi_1\Psi_{2,xx}\big)\\ &- \frac{1}{6}(r^2\Psi_1^2\Psi_{1,xxx} + r(r+3m-4)\Psi_1\Psi_{1,x}\Psi_{1,xx} + (m-1)(m-2)\Psi_{1,x}^3)\Big),\end{aligned} \tag{24}$$

$$\begin{aligned}\bar v_{m,0} =& -m\bar\Psi_{1,x},\\ \bar v_{m,1} =& m\Big(-\bar\Psi_{2,x} - \frac{1}{2}((2-r)\bar\Psi_1\bar\Psi_{1,xx} - (m+1)\bar\Psi_x^2\Big),\\ \bar v_{m,2} =& m\Big(-\bar\Psi_{3,x} - \frac{1}{2}\big((3-r)\bar\Psi_2\bar\Psi_{1,xx} - (2m+3)\Psi_{1,x}\Psi_{2,x} + (2-r)\bar\Psi_1\bar\Psi_{2,xx}\big)\\ &- \frac{1}{6}((2-r)^2\bar\Psi_1^2\bar\Psi_{1,xxx} + (2-r)(r+3m+2)\bar\Psi_1\bar\Psi_{1,x}\bar\Psi_{1,xx}\\ &+ (m+1)(m+2)\bar\Psi_{1,x}^3)\Big),\end{aligned} \tag{25}$$

and

$$\bar{u}_{m,j} = \begin{cases} \Big(\dfrac{\xi \circ X}{\xi}\Big)^{-\frac{m+1+j}{1-r}} \bar{v}_{m,j} \circ X, & \displaystyle\int_x^X \frac{\mathrm{d}x}{\xi(x)} = 1 - r, \quad r \neq 1, \\ \exp(-(m+1+j)\xi_x)\bar{v}_{m,j}, & r = 1. \end{cases}$$

Proof. To evaluate L^m we recall that L^m is connected to p^m through a canonical transformation by

$$L^m := \operatorname{Ad} \psi_< p^m = p^m + \{\Psi_<, p^m\} + \frac{1}{2}\{\Psi_<, \{\Psi_<, p^m\}\} + \frac{1}{6}\{\Psi_<, \{\Psi_<, \{\Psi_<, p^m\}\}\} + \cdots,$$

and compute the first Poisson brackets to obtain

$$\{\Psi_<, p^m\} = -m\Psi_{1,x}p^{m-1} - m\Psi_{2,x}p^{m-2} - m\Psi_{3,x}p^{m-3} + O(p^{m-4}),$$

$$\begin{aligned}\{\Psi_<, \{\Psi_<, p^m\}\} &= m\big(r\Psi_1\Psi_{1,xx} + (m-1)\Psi_{1,x}^2\big)p^{m-2} \\ &\quad + m\big((r+1)\Psi_2\Psi_{1,xx} + (2m-3)\Psi_{1,x}\Psi_{2,x} + r\Psi_1\Psi_{2,xx}\big)p^{m-3} + O(p^{m-4}),\end{aligned}$$

$$\begin{aligned}\{\Psi_<, \{\Psi_<, \{\Psi_<, p^m\}\}\} &= -m(r^2\Psi_1^2\Psi_{1,xxx} + r(r+3m-4)\Psi_1\Psi_{1,x}\Psi_{1,xx} \\ &\quad + (m-1)(m-2)\Psi_{1,x}^3)p^{m-3} + O(p^{m-4})\end{aligned}$$

when $p \to \infty$. Summing all terms and collecting those with the same power on p we get (24). For $\bar{\ell}$ we use

$$\bar{\ell}^m := \operatorname{Ad} \psi_> p^m = p^m + \{\Psi_>, p^m\} + \frac{1}{2}\{\Psi_>, \{\Psi_>, p^m\}\} + \frac{1}{6}\{\Psi_>, \{\Psi_>, \{\Psi_>, p^m\}\}\} + \cdots,$$

and we get formulae (25). An alternative way to deduce this is to use the intertwining transformation (15), $(p, x) \to (1/p, -x)$ together $r \to 2 - r$, that intertwines $\mathfrak{g}_>$ and $\mathfrak{g}_<$. Thus, the expressions for $\bar{v}_{m,j}$ are obtained from those for $u_{-m,j}$ by replacing $\partial_x^j \Psi_k$ by $(-1)^j \partial_x^j \bar{\Psi}_k$ and r by $2 - r$.

As $\bar{L}^m = \operatorname{Ad}_{\psi_{1-r}} \bar{\ell}^m$ and we have already obtained the expansion of $\bar{\ell}$ in powers of p, in order to compute $\bar{L}^m$, we need to characterize $\operatorname{Ad}_{\exp(\xi p^{1-r})}(\phi(x)p^n)$; i. e., to characterize the canonical transformation generated by ξp^{1-r}. For that aim is useful to perform the following calculation

$$\operatorname{ad}_{\xi p^{1-r}}(\phi(x)p^n) = \{\xi p^{1-r}, \phi(x)p^n\} = \big([(1-r)\xi\partial_x - n\xi_x](\phi)\big)p^n$$

so that

$$\operatorname{Ad}_{\exp(\xi p^{1-r})}(\phi(x)p^n) = [\exp((1-r)\xi\partial_x - n\xi_x)\phi(x)]p^n.$$

For $r = 1$; i.e., when $\mathfrak{g}_{1-r}$ is an Abelian Lie subalgebra, the action is easy to compute

$$\operatorname{Ad}_{\exp \xi}(\phi(x)p^n) = \exp(-n\xi_x)\phi(x)p^n.$$

However, for $r \neq 1$ the situation is rather more involved. Let us analyze this non-Abelian situation. The function

$$\Phi := \exp(\lambda((1-r)\xi\partial_x - n\xi_x)\phi(x)$$

is characterize –in a unique manner– by the following initial condition problem for a first-order linear PDE

$$\begin{gathered}\partial_\lambda \Phi = (1-r)\xi\partial_x\Phi - n\xi_x\Phi, \\ \Phi\big|_{\lambda=0} = \phi.\end{gathered}$$

The general solution of the PDE is

$$\Phi=g(\lambda+c(x))\xi^{\frac{n}{1-r}},\quad c(x):=\frac{1}{1-r}\int^x\frac{\mathrm{d}x}{\xi(x)}$$

where g is an arbitrary function, which is determined by the initial condition

$$\phi(x)=g(c(x))\xi^{\frac{n}{1-r}}.$$

A better characterization of g appears as follows: we first look for a function $X(x)$ defined implicitly by the relation

$$\int_x^X\frac{\mathrm{d}x}{\xi(x)}=(1-r)\lambda,$$

and hence

$$c(X)=\lambda+c(x).$$

Now, taking into account the initial condition we deduce

$$g(c(x))=\phi(x)\xi(x)^{\frac{n}{1-r}}$$

so that

$$g(c(X))=\phi(X)\xi(X)^{-\frac{n}{1-r}}$$

and, consequently,

$$\Phi(\lambda,x)=\phi(X(x))\Big(\frac{\xi(X(x))}{\xi(x)}\Big)^{-\frac{n}{1-r}}.$$

Hence

$$\mathrm{Ad}_{\exp(\xi p^{1-r})}(\phi(x)p^n)=\begin{cases}\Big(\dfrac{\xi(X(x))}{\xi(x)}\Big)^{-\frac{n}{1-r}}\phi(X(x))p^n, & \displaystyle\int_x^X\frac{\mathrm{d}x}{\xi(x)}=(1-r),\quad r\neq 1,\\ \exp(-n\xi_x(x))\phi(x)p^n, & r=1.\end{cases}\tag{26}$$

$\square$

Observe that from the proof of the above Proposition 1 we deduce that $u_{m,j}=-m\Psi_{j+1,x}+U_{m,j}$ where $U_{m,j}$ is a nonlinear function of $\Psi_1,\dots,\Psi_j$ and its x-derivatives.

To give the Lax equations (19) the Zakharov–Shabat form; i. e., a zero-curvature representation we introduce the the exterior differential with respect to the variables $\{t_n,\bar t_n\}_{n\geq 1}$

$$\mathrm{d}:=\sum_{n\geq 1}\Big(\frac{\partial}{\partial t_n}\mathrm{d}t_n+\frac{\partial}{\partial\bar t_n}\mathrm{d}\bar t_n\Big).$$

Then, the factorization problem (4) implies

$$\mathrm{d}\psi_<\cdot\psi_<^{-1}+\mathrm{Ad}_{\psi_<}\mathrm{d}t=\mathrm{d}\psi_\geqslant\cdot\psi_\geqslant^{-1}+\mathrm{Ad}_{\psi_\geqslant}\mathrm{d}\bar t.\tag{27}$$

and, if we define

$$\Omega:=\mathrm{Ad}_{\psi_<}\mathrm{d}t-\mathrm{Ad}_{\psi_\geqslant}\mathrm{d}\bar t=\sum_{n\geq 1}(L^{n+1-r}\mathrm{d}t_n-\bar L^{1-r-n}\mathrm{d}\bar t_n),$$

we may rewrite (27) as follows

$$\Omega = \mathrm{d}\psi_{\geqslant} \cdot \psi_{\geqslant}^{-1} - \mathrm{d}\psi_{<} \cdot \psi_{<}^{-1}. \tag{28}$$

Equation (28) implies

$$\mathrm{d}\psi_{<} \cdot \psi_{<}^{-1} = -P_{<}\Omega =: -\Omega_{<}, \quad \mathrm{d}\psi_{\geqslant} \cdot \psi_{\geqslant}^{-1} = P_{\geqslant}\Omega =: \Omega_{\geqslant},$$

that coincides with (18) when splitted in coordinates. Hence, we deduce the following zero-curvature conditions

$$\mathrm{d}\Omega_{<} = -\{\Omega_{<}, \Omega_{<}\}, \quad \mathrm{d}\Omega_{\geqslant} = \{\Omega_{\geqslant}, \Omega_{\geqslant}\}.$$

3 The associated dispersionless integrable hierarchies

We now deduce the integrable hierarchies associated with the factorization problem (4), namely the r-th dispersionless modified KP, r-th dispersionless Dym and r-th dispersionless Toda hierarchies.

3.1 The r-th dispersionless modified KP hierarchy

We will study now the consequences of (9) and derive a nonlinear PDE for $u_{1,0}$ which resemble the modified dispersionless KP equation, and was found in [1]. For the sake of simplicity we write (9) as

$$\partial_n\psi_{<} \cdot \psi_{<}^{-1} + P_{<}L^{n+1-r} = 0.$$

The right derivatives of $\psi_{<}$, as follows from (6), are

$$\partial_n\psi_{<} \cdot \psi_{<}^{-1} = \partial_n\Psi_{<} + \frac{1}{2}\{\Psi_{<}, \partial_n\Psi_{<}\} + \frac{1}{6}\{\Psi_{<}, \{\Psi_{<}, \partial_n\Psi_{<}\}\} + \cdots,$$

so that

$$\partial_n\psi_{<} \cdot \psi_{<}^{-1} = (\partial_n\Psi_1)p^{-r} + \big(\partial_n\Psi_2 + \frac{r}{2}(\Psi_{1,x}\partial_n\Psi_1 - \Psi_1\partial_n\Psi_{1,x})\big)p^{-r-1} + O(p^{-r-2}) \tag{29}$$

for $p \to \infty$. Recall that when $p \to \infty$ we have

$$L^{n+1-r} = p^{n+1-r} + u_{n+1-r,0}p^{n-r} + u_{n+1-r,1}p^{n-r-1} + u_{n+1-r,2}p^{n-r-2} + O(p^{n-r-3})$$

with $u_{n+1-r,j} = -(n+1-r)\Psi_{j+1,x} + U_{n+1-r,j}$ and $U_{n+1-r,j}$ a given nonlinear function of $\Psi_1,\dots,\Psi_j$ and its x-derivatives. Equation (9) together with (29) gives an infinite set of equations, among which the two first are

$$\begin{aligned} \partial_n\Psi_1 &= -(n+1-r)\Psi_{n+1,x} + U_{n+1-r,n}, \\ \partial_n\Psi_2 + \frac{r}{2}(\Psi_{1,x}\partial_n\Psi_1 - \Psi_1\partial_n\Psi_{1,x}) &= -(n+1-r)\Psi_{n+2,x} + U_{n+1-r,n+1} \end{aligned}$$

Thus, we get for $\Psi_{n+j,x}$, $j = 1, 2, \dots$ expressions in terms of $\Psi_1, \dots, \Psi_n$ together with its x-derivatives and integrals and also its ∂_n-derivative. For the next flow we have

$$\begin{aligned} \partial_{n+1}\Psi_1 &= -(n+2-r)\Psi_{n+2,x} + U_{n+2-r,n+1}, \\ \partial_{n+1}\Psi_2 + \frac{r}{2}(\Psi_{1,x}\partial_{n+1}\Psi_1 - \Psi_1\partial_{n+1}\Psi_{1,x}) &= -(n+2-r)\Psi_{n+3,x} + U_{n+3-r,n+2} \end{aligned}$$

from where it follows a nonlinear PDE system for $(\Psi_1,\dots,\Psi_n)$, in the variables x,t_n,t_{n+1}.

In particular, if $r\neq 2$ —when $r=2$ the t_1-flow is trivial— and $n=1,2$ we get

$$\Psi_{2,x}=\frac{1}{2-r}\partial_1\Psi_1+\frac{1}{2}\big(r\Psi_1\Psi_{1,xx}+(1-r)\Psi_{1,x}^2\big), \tag{30}$$

$$\begin{aligned}\partial_2\Psi_1=&\frac{3-r}{2-r}\Big(\partial_1\Psi_2+\frac{r}{2}\big(\Psi_{1,x}\partial_1\Psi_1-\Psi_1\partial_1\Psi_{1,x}\big)\Big)\\&+(3-r)\Big(-\Psi_{1,x}\Psi_{2,x}+\frac{r}{2}\Psi_1\Psi_{1,x}\Psi_{1,xx}-\frac{r-1}{3}\Psi_{1,x}^3\Big),\end{aligned} \tag{31}$$

and hence,

$$\begin{aligned}\Psi_{1,xt_2}=&\frac{3-r}{(2-r)^2}\Psi_{1,t_1t_1}-\frac{(3-r)(1-r)}{2-r}\Psi_{1,xx}\Psi_{1,t_1}\\&-\frac{(3-r)r}{2-r}\Psi_{1,x}\Psi_{1,xt_1}-\frac{(3-r)(1-r)}{2}\Psi_{1,x}^2\Psi_{1,xx}.\end{aligned} \tag{32}$$

If we introduce

$$u:=u_{1,0}=-\Psi_{1,x},\quad \partial_x^{-1}u=\int_{x_0}^x u(x)\mathrm{d}x \tag{33}$$

we get for u the following nonlinear PDE,

$$u_{t_2}=\frac{3-r}{(2-r)^2}(\partial_x^{-1}u)_{t_1t_1}+\frac{(3-r)(1-r)}{2-r}u_x(\partial_x^{-1}u)_{t_1}+\frac{r(3-r)}{2-r}uu_{t_1}-\frac{(3-r)(1-r)}{2}u^2u_x. \tag{34}$$

This equation is similar to the dispersionless modified KP equation which is recovered for $r=0$, hence we called it r-th dispersionless modified KP (r-dmKP) equation, note that it was derived for the first time in [1]. Therefore we will refer to equation (32) as the potential r-dmKP equation.

The $\bar t_n$ flows for $\psi_>$ From the intertwining property we find out that the $\bar t_n$-flows for $\bar\Psi_k$ are derived from the t_n-flows for Ψ_k by replacing r by $2-r$, and each $\partial_x^j\Psi_k$ by $(-1)^j\partial_x^j\bar\Psi_k$ so that (32) goes to

$$\begin{aligned}-\bar\Psi_{1,x\bar t_2}=&\frac{1+r}{r^2}\bar\Psi_{1,\bar t_1\bar t_1}+\frac{(1+r)(1-r)}{r}\bar\Psi_{1,xx}\bar\Psi_{1,\bar t_1}\\&-\frac{(2-r)(1+r)}{r}\bar\Psi_{1,x}\bar\Psi_{1,x\bar t_1}+\frac{(1+r)(1-r)}{2}\bar\Psi_{1,x}^2\bar\Psi_{1,xx}\end{aligned}$$

or, in terms of $\bar v:=\bar v_1=-\bar\Psi_{1,x}$,

$$-\bar v_{\bar t_2}=\frac{1+r}{r^2}\partial_x^{-1}\bar v_{\bar t_1\bar t_1}-\frac{(1+r)(1-r)}{r}\bar v_x\partial_x^{-1}\bar v_{\bar t_1}+\frac{(2-r)(1+r)}{r}\bar v\bar v_{\bar t_1}+\frac{(1+r)(1-r)}{2}\bar v^2\bar v_x.$$

3.2 The r-th dispersionless Dym hierarchy

Here we shall discuss the consequences of the equations (10) and (13). In the one hand, we may rewrite (10) as

$$P_{1-r}L^{n+1-r} = \partial_n\psi_{1-r}\cdot\psi_{1-r}^{-1},$$

which in terms of $\bar{\psi}_{1-r} := \psi_{1-r}^{-1} = \exp(-\xi p^{1-r})$ reads as

$$\partial_n\bar{\psi}_{1-r}\cdot\bar{\psi}_{1-r}^{-1} + \mathrm{Ad}_{\bar{\psi}_{1-r}}(u_{n+1-r,n-1}p^{1-r}) = 0. \tag{35}$$

On the other hand (13) reads

$$\bar{\partial}_n\psi_{1-r}\cdot\psi_{1-r}^{-1} + \mathrm{Ad}_{\psi_{1-r}}(\bar{v}_{1-r-n,n-1}p^{1-r}) = 0. \tag{36}$$

At this point is useful to recall (26) which reads

$$\mathrm{Ad}_{\exp(\xi p^{1-r})}(f(x)p^n) = \begin{cases} X_x(x)^{-\frac{n}{1-r}}f(X(x))p^n, & \displaystyle\int_x^X \frac{\mathrm{d}x}{\xi(x)} = (1-r), \quad r\neq 1,\\ \exp(-n\xi_x(x))f(x)p^n, & \qquad\qquad\qquad\qquad r=1,\end{cases}$$

where have used

$$X_x = \frac{\xi(X(x))}{\xi(x)}.$$

In particular (26) implies for $r\neq 1$ the following equations

$$\begin{aligned}\mathrm{Ad}_{\exp(\xi p^{1-r})}(x) &= X, & \int_x^X \frac{\mathrm{d}x}{\xi(x)} &= (1-r),\\ \mathrm{Ad}_{\exp(-\xi p^{1-r})}(x) &= \bar{X}, & \int_x^{\bar{X}} \frac{\mathrm{d}x}{\xi(x)} &= -(1-r),\end{aligned}$$

Observe that $\bar{X}$ is the inverse function of X; i.e., $\bar{X}\circ X = \mathrm{id}$. This is also a consequence $x = \mathrm{Ad}_{\exp(-\xi p^{1-r})}(X(x)) = \bar{X}(X(x))$. Another remarkable fact is that X is the canonical transform of the x variable under ψ_{1-r}. Note also that the conjugate momenta to the variables X and $\bar{X}$ are

$$\mathrm{Ad}_{\exp(\xi p^{1-r})}(p) = (X_x)^{-\frac{1}{1-r}}p, \quad \mathrm{Ad}_{\exp(-\xi p^{1-r})}(p) = (\bar{X}_x)^{-\frac{1}{1-r}}p,$$

respectively. Finally, notice that when $r=1$ we have $X(x)=\bar{X}(x)=x$.

Now, if ∂ is a given Lie algebra derivation then

$$\begin{aligned}\partial\psi_{1-r}\cdot\psi_{1-r}^{-1} &= \beta(x)p^{1-r},\\ \partial\bar{\psi}_{1-r}\cdot\bar{\psi}_{1-r}^{-1} &= \bar{\beta}(x)p^{1-r},\end{aligned} \tag{37}$$

but

$$\partial\psi_{1-r}\cdot\psi_{1-r}^{-1} + \mathrm{Ad}_{\psi_{1-r}}(\partial\bar{\psi}_{1-r}\cdot\bar{\psi}_{1-r}^{-1}) = 0 \Rightarrow \beta(x)p^{1-r} + \mathrm{Ad}_{\psi_{1-r}}(\bar{\beta}(x)p^{1-r}) = 0,$$

so that

$$\beta(x) + \bar{\beta}(X(x))X_x(x)^{-1} = 0,$$

or, upon the use of $X_x(\bar{X}(x))\bar{X}_x = 1$,

$$\beta(\bar{X}(x)) + \bar{\beta}(x)\bar{X}_x(x) = 0. \tag{38}$$

From (37) we get

$$\begin{aligned}
\partial X &= \{\beta(x)p^{1-r}, X(x)\} = (1-r)\beta X_x,\\
\partial \bar{X} &= \{\bar{\beta}(x)p^{1-r}, \bar{X}(x)\} = (1-r)\bar{\beta}\bar{X}_x,
\end{aligned}$$

so that, when $r \neq 1$,

$$\beta = \frac{1}{1-r}\frac{\partial X}{X_x}, \quad \bar{\beta} = \frac{1}{1-r}\frac{\partial \bar{X}}{\bar{X}_x}.$$

Observe that the compatibility with (38) follows from that $\partial X(\bar{X}(x)) + X_x(\bar{X})\partial\bar{X}(\bar{X}(x)) = 0$.

Therefore, we have proven that

$$\partial\psi_{1-r}\cdot\psi_{1-r}^{-1} = \begin{cases} \dfrac{1}{1-r}\dfrac{\partial X}{X_x}p^{1-r}, & \displaystyle\int_x^X \frac{\mathrm{d}x}{\xi(x)} = (1-r), \quad r \neq 0,\\ \partial\xi\, p^{1-r}, & \qquad\qquad\qquad\qquad\qquad r = 1,\end{cases}$$

$$\partial\bar{\psi}_{1-r}\cdot\bar{\psi}_{1-r}^{-1} = \begin{cases} \dfrac{1}{1-r}\dfrac{\partial \bar{X}}{\bar{X}_x}p^{1-r}, & \displaystyle\int_x^{\bar{X}} \frac{\mathrm{d}x}{\xi(x)} = -(1-r), \quad r \neq 0,\\ -\partial\xi\, p^{1-r}, & \qquad\qquad\qquad\qquad\qquad r = 1.\end{cases}$$

We are now ready to tackle (35) and (36), which read

$$\begin{cases} \partial_n\bar{X} = -(1-r)u_{n+1-r,n-1}(\bar{X}), & r \neq 1,\\ \partial_n\xi = u_{n,n-1}, & r = 1,\end{cases} \tag{39}$$

$$\begin{cases} \bar{\partial}_n X = -(1-r)\bar{v}_{1-r-n,n-1}(X), & r \neq 1,\\ \bar{\partial}_n\xi = -\bar{v}_{-n,n-1}, & r = 1.\end{cases} \tag{40}$$

Let us look at the consequences of (39) for $n = 1$ and $n = 2$ and recall (24):

$$\begin{cases} \bar{X}_{t_1} = (1-r)(2-r)\Psi_{1,x}(\bar{X}), & r \neq 1\\ \xi_{t_1} = -\Psi_{1,x}, & r = 1,\end{cases} \tag{41}$$

$$\begin{cases} \begin{aligned}\bar{X}_{t_2} = -(1-r)(3-r)\Big(&\Psi_{2,x}(\bar{X})\\ &-\frac{1}{2}(r\Psi_1(\bar{X})\Psi_{1,xx}(\bar{X}) + (2-r)\Psi_{1,x}(\bar{X})^2)\Big),\end{aligned} & r \neq 1,\\ \xi_{t_2} = -2(\Psi_{2,x} - \frac{1}{2}(\Psi_1\Psi_{1,xx} + \Psi_{1,x}^2)), & r = 1.\end{cases} \tag{42}$$

We first analyze the case $r \neq 1$. The first equation (41), when $r \neq 1, 2$, gives the important relation

$$\Psi_{1,x}(\bar{X}) = \frac{1}{(1-r)(2-r)}\bar{X}_{t_1}. \tag{43}$$

By introducing (30) into (42) we get

$$\bar{X}_{t_2} = (1-r)(3-r)\Big(\frac{1}{2-r}\Psi_{1,t_1}(\bar{X}) - \frac{1}{2}\Psi_{1,x}(\bar{X})^2\Big),$$

which we are going to manipulate. Firstly, we take its x derivative

$$\bar{X}_{xt_2} = (1-r)(3-r)\Big(\frac{1}{2-r}\Psi_{1,t_1x}(\bar{X})\bar{X}_x - \Psi_{1,x}(\bar{X})(\Psi_{1,x}(\bar{X}))_x\Big),$$

second we see that

$$\begin{aligned}\Psi_{1,t_1x}(\bar{X})\bar{X}_x =&\big((\Psi_{1,x}(\bar{X}))_{t_1} - \Psi_{1,xx}(\bar{X})\bar{X}_{t_1}\big)\bar{X}_x\\ =&(\Psi_{1,x}(\bar{X}))_{t_1}\bar{X}_x - (\Psi_{1,x}(\bar{X}))_x\bar{X}_{t_1}.\end{aligned}$$

Therefore,

$$\bar{X}_{xt_2} = (1-r)(3-r)\Big(\frac{1}{2-r}\big((\Psi_{1,x}(\bar{X}))_{t_1}\bar{X}_x - (\Psi_{1,x}(\bar{X}))_x\bar{X}_{t_1}\big) - \Psi_{1,x}(\bar{X})(\Psi_{1,x}(\bar{X}))_x\Big),$$

that recalling (43) reads as follows

$$\bar{X}_{xt_2} = \frac{3-r}{2-r}\Big(\frac{1}{2-r}\bar{X}_{t_1t_1}\bar{X}_x - \frac{1}{1-r}\bar{X}_{xt_1}\bar{X}_{t_1}\Big) \tag{44}$$

In the case $r=1$ we introduce (30) into (42) to get

$$\xi_{t_2} = -2\Psi_{1,t_1} + \Psi_{1,x}^2,$$

that taking the x-derivative reads

$$\xi_{xt_2} = -2\Psi_{1,xt_1} + 2\Psi_{1,x}\Psi_{1,xx},$$

and recalling (41) for $r=1$ is

$$\xi_{t_2x} - 2\xi_{t_1}\xi_{t_1x} - 2\xi_{t_1t_1} = 0, \tag{45}$$

Upon the introduction of the variable

$$v = \begin{cases}(\bar{X}_x)^{-\frac{1}{1-r}}, & r\neq 1,\\ \exp\xi_x, & r=1,\end{cases} \tag{46}$$

equations (44) and (45) transforms onto

$$v_{t_2} = \frac{3-r}{(2-r)^2}v^{r-1}\Big(v^{2-r}\partial_x^{-1}(v^{r-2}v_{t_1})\Big)_{t_1} \tag{47}$$

which resembles the dispersionless Dym equation which appears for $r=0$, hence we refer to it as the r-th dispersionless Dym (r-dDym) equation (for $r\neq 2$); this equation was first derived in [1]. We shall refer to (44) as the potential r-dDym equation.

From (40) we derive

$$\begin{cases} X_{\bar{t}_1} = -(1-r)r\bar{\Psi}_{1,x}(X), & r\neq 1\\ \xi_{\bar{t}_1} = -\bar{\Psi}_{1,x}, & r=1\end{cases} \tag{48}$$

$$\begin{cases} X_{\bar{t}_2} = -(1-r)(1+r)\Big(\bar{\Psi}_{2,x}(X) + \frac{1}{2}((2-r)\bar{\Psi}_1(X)\bar{\Psi}_{1,xx}(X) + r\bar{\Psi}_{1,x}(X)^2)\Big), & r\neq 1,\\ \xi_{\bar{t}_2} = -2(\bar{\Psi}_{2,x} + \frac{1}{2}(\bar{\Psi}_1\bar{\Psi}_{1,xx} + \bar{\Psi}_{1,x}^2)), & r=1,\end{cases} \tag{49}$$

which, for $r \neq 1$, lead to

$$-X_{x\bar{t}_2} = \frac{1+r}{r}\Big(\frac{1}{r}X_{\bar{t}_1\bar{t}_1}X_x + \frac{1}{1-r}X_{x\bar{t}_1}X_{\bar{t}_1}\Big), \tag{50}$$

and when $r = 1$ to

$$\xi_{\bar{t}_2x} + 2\xi_{\bar{t}_1}\xi_{\bar{t}_1x} + 2\xi_{\bar{t}_1\bar{t}_1} = 0. \tag{51}$$

Observe that equations (40) can be obtained from (39) with the use of the intertwining (15) in the following two steps:

1. We have
$$\partial'_n\bar{X}' = -(1-r')u'_{n'+1-r',n'-1}(\bar{X}'),$$
that recalling that $\partial'_n = \bar{\partial}_n$, $r' = 2-r$ and $u'_{n'+1-r',n'-1}(\bar{X}') = \bar{v}_{1-r-n,n-1}(-\bar{X}')$ $(x' = -x)$ reads
$$\bar{\partial}_n\bar{X}' = (1-r)\bar{v}_{1-r-n,n-1}(-\bar{X}').$$

2. Now, we only need to find $\bar{X}'$; this can be done in at least two ways. From the definition we have
$$\bar{X}' = \mathrm{Ad}_{\psi'^{-1}_{1-r'}}(x') = \mathrm{Ad}_{\psi_{1-r}}(-x) = -X,$$
an alternative is from
$$\int_{x'}^{\bar{X}'}\frac{\mathrm{d}x}{\xi'(x)} = -(1-r') \Rightarrow \int_{-x}^{\bar{X}'}\frac{\mathrm{d}x}{-\xi(-x)} = (1-r) \Rightarrow \int_{x}^{-\bar{X}'}\frac{\mathrm{d}x}{\xi(x)} = (1-r).$$
So that $\bar{X}' = -X$.

3. For $r = 1$ we only need to recall that $\psi_0'^{-1} = \psi_0$ so that $\xi'(x') = -\xi(-x)$.

3.3 Miura map among r-dmKP and r-dDym equations

From (33), (43) and (46) we get

$$\begin{cases} -\dfrac{1}{(1-r)(2-r)}\bar{X}_{t_1} = u(\bar{X}), & \bar{X} = \partial_x^{-1}v^{r-1}, \quad r \neq 1, \\ \xi_{t_1} = u, & \xi = \partial_x^{-1}\log v, \quad r = 1 \end{cases}$$

and

$$\begin{cases} \partial_x^{-1}(v^{r-1})_{t_1} = -(2-r)(1-r)u(\partial_x^{-1}v^{r-1}), & r \neq 1 \\ \partial_x^{-1}(\log v)_{t_1} = u(x), & r = 1. \end{cases} \tag{52}$$

Equations (52) relate solutions u and v of the r-dmKP (34) and r-dDym (47) equations.

If we derive with respect to x the above relations we get

$$\begin{cases} u_x(\partial_x^{-1}v^{r-1}) = \dfrac{1}{2-r}(\log v)_{t_1}, & r \neq 1, \\ u_x = (\log v)_{t_1}, & r = 1. \end{cases} \tag{53}$$

In any case observe that a solution v to the r-dDym equation provide us with a solution of the r-dmKP equation, after the calculation of some inverse functions of $\bar{X}$, but the reverse, given a solution u of the r-dmKP equation (34) to get v a solution of the r-dDym equation (47) do not follow from formulae either (52) neither (53). Observe that in [3] a similar Miura map was derived, in a quite different manner, for the well known $r = 0$ case.

3.4 The r-th dispersionless Toda hierarchy

Here we consider equations (11) and (12)

$$\operatorname{Ad}_{\bar\psi_{1-r}} P_{>} L^{n+1-r} = \partial_n \psi_{>} \cdot \psi_{>}^{-1},$$
$$\operatorname{Ad}_{\psi_{1-r}} P_{<} \bar\ell^{1-r-n} = \bar\partial_n \psi_{<} \cdot \psi_{<}^{-1},$$

that for $n=1$ reads

$$\operatorname{Ad}_{\bar\psi_{1-r}} P_{>} L^{2-r} = \partial_1 \psi_{>} \cdot \psi_{>}^{-1}, \tag{54}$$
$$\operatorname{Ad}_{\psi_{1-r}} P_{<} \bar\ell^{-r} = \bar\partial_1 \psi_{<} \cdot \psi_{<}^{-1}. \tag{55}$$

Looking at the leading terms in p we obtain from (54) and (55) the following equations

$$\bar\Psi_{1,t_1} = \begin{cases} (\bar X_x)^{-\frac{2-r}{1-r}}, & r \neq 1, \\ \exp(\xi_x), & r = 1, \end{cases}$$
$$\Psi_{1,\bar t_1} = \begin{cases} (X_x)^{\frac{r}{1-r}}, & r \neq 1, \\ \exp(\xi_x), & r = 1. \end{cases}$$

Recall now equations (43) and (48) which, taking into account $\bar X_{t_1}(X) = -X_{t_1}(x)/X_x(x)$ and $X_{\bar t_1}(\bar X) = -\bar X_{\bar t_1}(x)/\bar X_x(x)$, can be written as

$$\Psi_{1,x} = \begin{cases} -\dfrac{1}{(1-r)(2-r)}\dfrac{X_{t_1}}{X_x}, & r \neq 1, \\ -\xi_{t_1}, & r = 1, \end{cases}$$
$$\bar\Psi_{1,x} = \begin{cases} \dfrac{1}{(1-r)r}\dfrac{\bar X_{\bar t_1}}{\bar X_x}, & r \neq 1, \\ -\xi_{\bar t_1}, & r = 1, \end{cases}$$

respectively. The compatibility of these equations lead to

$$\Big((\bar X_x)^{-\frac{2-r}{1-r}}\Big)_x - \frac{1}{(1-r)r}\Big(\frac{\bar X_{\bar t_1}}{\bar X_x}\Big)_{t_1} = 0, \tag{56a}$$
$$\Big((X_x)^{\frac{r}{1-r}}\Big)_x + \frac{1}{(1-r)(2-r)}\Big(\frac{X_{t_1}}{X_x}\Big)_{\bar t_1} = 0, \tag{56b}$$

when $r \neq 1$ while for $r=1$ the equation is

$$(\exp(\xi_x))_x + \xi_{t_1\bar t_1} = 0, \tag{57}$$

this are new integrable equations, which we call r-th dispersionles Toda (r-dToda) equation, because for $r=1$ the corresponding equation is the dispersionless Toda equation —known also as the Boyer–Finley equation—.

Equations (56a) and (56b) are the same equation, indeed. To prove it we just need to evaluate equation (56b) on $\bar X$ and recall that X is the inverse function of $\bar X$, $X(\bar X(x)) = x$, so that

$$X_x(\bar X(x))\bar X_x(x) = 1,$$
$$X_{t_1}(\bar X(x)) + X_x(\bar X(x))\bar X_{t_1}(x) = 0.$$

From the relations

$$\frac{X_{t_1}(\bar{X}(x))}{X_x(\bar{X}(x))} = -\bar{X}_{t_1},$$

$$\Big(\frac{X_{t_1}}{X_x}\Big)_x(\bar{X}) = \Big(\frac{X_{t_1}(\bar{X})}{X_x(\bar{X})}\Big)_x\frac{1}{\bar{X}_x},$$

$$\Big(\frac{X_{t_1}}{X_x}\Big)_{\bar{t}_1}(\bar{X}) = \Big(\frac{X_{t_1}(\bar{X})}{X_x(\bar{X})}\Big)_{\bar{t}_1} - \Big(\frac{X_{t_1}}{X_x}\Big)_x(\bar{X})\bar{X}_{\bar{t}_1}$$

we derive

$$\Big(\frac{X_{t_1}}{X_x}\Big)_{\bar{t}_1}(\bar{X}) = -\bar{X}_{t_1\bar{t}_1} + \frac{\bar{X}_{xt_1}\bar{X}_{\bar{t}_1}}{\bar{X}_x} = -\bar{X}_x\Big(\frac{\bar{X}_{\bar{t}_1}}{\bar{X}_x}\Big)_{t_1}.$$

We evaluate now

$$\big((X_x)^{\frac{r}{1-r}}\big)_x(\bar{X}) = \big((X_x(\bar{X}))^{\frac{r}{1-r}}\big)_x\frac{1}{\bar{X}_x} = \big((\bar{X}_x)^{-\frac{r}{1-r}}\big)_x\frac{1}{\bar{X}_x}.$$

Therefore, we conclude that (56b) imply

$$\big((\bar{X}_x)^{-\frac{r}{1-r}}\big)_x\frac{1}{\bar{X}_x^2} - \frac{1}{(1-r)(2-r)}\Big(\frac{\bar{X}_{\bar{t}_1}}{\bar{X}_x}\Big)_{t_1} = 0$$

but observing

$$\big((\bar{X}_x)^{-\frac{r}{1-r}}\big)_x\frac{1}{\bar{X}_x^2} = \frac{r}{2-r}\big((\bar{X}_x)^{-\frac{2-r}{1-r}}\big)_x$$

we deduce, as claimed, equation (56a).

4 Additional symmetries

In this section we deal with the additional symmetries of the integrable hierarchies just described. We first introduced the Orlov functions $M, \bar{M}, \bar{m}$ in this context and the consider the construction of additional symmetries. We compute explicitly some of these additional symmetries for the potential r-dmKP (32), the r-dDym (44) and the r-dToda (56a) equations, finding explicit symmetries of these nonlinear equations depending on arbitrary functions of the variable t_2.

4.1 The Orlov funtions

In formulae (17) we introduced the Lax functions $L, \bar{\ell}$ and $\bar{L}$, which are the canonical transformation of the p variable through $\psi_<, \psi_>$ and $\psi_\geqslant$, respectively. Recalling that t and $\bar{t}$ are functions of p only we can write these Lax functions as follows:

$$L = \operatorname{Ad}_{\psi_<\cdot\exp t}p, \qquad \bar{\ell} = \operatorname{Ad}_{\psi_>\cdot\exp\bar{t}}p, \qquad \bar{L} = \operatorname{Ad}_{\psi_\geqslant\cdot\exp\bar{t}}p.$$

The Orlov functions $M, \bar{m}$ and $\bar{M}$ are defined analogously with the replacement of p by x:

$$M := \operatorname{Ad}_{\psi_<\cdot\exp t}x, \quad \bar{m} := \operatorname{Ad}_{\psi_>\cdot\exp\bar{t}}x, \quad \bar{M} := \operatorname{Ad}_{\psi_\geqslant\cdot\exp\bar{t}}x. \tag{58}$$

In the next Proposition we describe the form of the Orlov functions as series in the Lax functions.

Proposition 2. *The Orlov functions defined in* (58) *have the following expansions*

$$
\begin{aligned}
M &= \cdots + w_2 L^{-2} + w_1 L^{-1} + x + (2-r)t_1 L + (3-r)t_2 L^2 + \cdots, \quad L \to \infty\\
\bar{m} &= \cdots - (r+1)\bar{t}_2 \bar{\ell}^{-2} - r\bar{t}_1 \bar{\ell}^{-1} + x + \bar{\omega}_1 \bar{\ell} + \bar{\omega}_2 \bar{\ell}^2 + \cdots, \quad \bar{\ell} \to 0\\
\bar{M} &= \cdots - (r+1)\bar{t}_2 \bar{L}^{-2} - r\bar{t}_1 \bar{L}^{-1} + X + \bar{w}_1 \bar{L} + \bar{w}_2 \bar{L}^2 + \cdots, \quad \bar{L} \to 0
\end{aligned}
$$

with

$$
\begin{aligned}
w_1 &= -r\Psi_1,\\
w_2 &= -(r+1)\big(\Psi_2 - \frac{1}{2} r \Psi_1 \Psi_{1,x}\big),\\
&\vdots\\
\bar{\omega}_1 &= (2-r)\bar{\Psi}_1(x),\\
\bar{\omega}_2 &= (3-r)\big(\bar{\Psi}_2(x) - \frac{1}{2}(2-r)\bar{\Psi}_1(x)\bar{\Psi}_{1,x}(x)\big),\\
&\vdots
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{w}_1 &= (2-r)\bar{\Psi}_1(X),\\
\bar{w}_2 &= (3-r)\big(\bar{\Psi}_2(X) - \frac{1}{2}(2-r)\bar{\Psi}_1(X)\bar{\Psi}_{1,x}(X)\big),\\
&\vdots
\end{aligned}
$$

Proof. Now, taking into account that

$$
\operatorname{ad}_t x = \{t, x\} = p^r \frac{\partial t}{\partial p} = (2-r)t_1 p + (3-r)t_2 p^2 + \cdots
$$

we evaluate

$$
\operatorname{Ad}_{\exp t} x = \exp(\operatorname{ad}_t)(x) = x + p^r \frac{\partial t}{\partial p} = x + (2-r)t_1 p + (3-r)t_2 p^2 + \cdots,
$$

and, therefore,

$$
\begin{aligned}
M &= \operatorname{Ad}_{\psi_<}\big(x + p^r \frac{\partial t}{\partial p}\big) = \operatorname{Ad}_{\psi_<}(x) + L^r \frac{\partial t(L)}{\partial L}\\
&= \operatorname{Ad}_{\psi_<}(x) + (2-r)t_1 L + (3-r)t_2 L^2 + \cdots.
\end{aligned}
$$

To compute M we need to evaluate

$$
\operatorname{ad}_{\Psi_<}(x) = \{\Psi_<, x\} = p^r \frac{\partial \Psi_<}{\partial p} = D_p \Psi_<
$$

where

$$
D_p := p^r \frac{\partial}{\partial p}.
$$

Notice that D_p is a derivation of the Lie algebra $\mathfrak{g}$:

$$
D_p\{f, g\} = \{D_p f, g\} + \{f, D_p g\}.
$$

Thus

$$\mathrm{Ad}_{\psi_<}(x) = x + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \,\mathrm{ad}^n_{\Psi_<} D_p\Psi_< = x + D_p\psi_< \cdot \psi_<^{-1},$$

where

$$D_p\Psi_< = -(r\Psi_1 p^{-1} + (r+1)\Psi_2 p^{-2} + \cdots).$$

We can compute now $D_p\psi_< \cdot \psi_<^{-1}$:

$$\begin{aligned} D_p\psi_< \cdot \psi_<^{-1} &= D_p\Psi_< + \frac{1}{2}\{\Psi_<, D_p\Psi_<\} + \cdots \\ &= -r\Psi_1 p^{-1} - \big((r+1)\Psi_2 - \frac{1}{2}r(r-1)\Psi_1\Psi_{1,x}\big)p^{-2} + \cdots . \end{aligned}$$

From

$$L^m = p^m + u_{m,0}p^{m-1} + u_{m,1}p^{m-2} + \cdots$$

we deduce

$$p^m = L^m - u_{m,0}L^{m-1} - (u_{m,1} - u_{m-1,1}u_{m,0})L^{m-2} - \cdots$$

so that

$$M = \cdots + w_2 L^{-2} + w_1 L^{-1} + x + (2-r)t_1 L + (3-r)t_2 L^2 + \cdots , \tag{59}$$

where, for example

$$\begin{aligned} w_1 &= -r\Psi_1, \\ w_2 &= -(r+1)\big(\Psi_2 - \frac{1}{2}r\Psi_1\Psi_{1,x}\big). \end{aligned}$$

For $\bar{m}$ and $\bar{M}$ we proceed in a similar manner. First

$$\begin{aligned} \mathrm{ad}_{\bar{t}}\, x &= \{\bar{t}, x\} = p^r \frac{\partial \bar{t}}{\partial p} = -r\bar{t}_1 p^{-1} - (r+1)\bar{t}_2 p^{-2} + \cdots \\ \mathrm{Ad}_{\exp \bar{t}}\, x &= x + p^r \frac{\partial \bar{t}}{\partial p} = x - r\bar{t}_1 p^{-1} - (r+1)\bar{t}_2 p^{-2} + \cdots , \end{aligned}$$

so that

$$\begin{aligned} \bar{m} &= \mathrm{Ad}_{\psi_>}(x) + \bar{\ell}^r \frac{\partial \bar{t}(\bar{\ell})}{\partial \bar{\ell}} \\ &= \mathrm{Ad}_{\psi_>}(x) - r\bar{t}_1 \bar{\ell}^{-1} - (r+1)\bar{t}_2 \bar{\ell}^{-2} + \cdots , \\ \bar{M} &= \mathrm{Ad}_{\psi_\geqslant}(x) + \bar{L}^r \frac{\partial t(\bar{L})}{\partial \bar{L}} \\ &= \mathrm{Ad}_{\psi_\geqslant}(x) - r\bar{t}_1 \bar{L}^{-1} - (r+1)\bar{t}_2 \bar{L}^{-2} + \cdots . \end{aligned}$$

Now

$$\mathrm{Ad}_{\psi_>}(x) = x + D_p\psi_> \cdot \psi_>^{-1}.$$

and

$$D_p\Psi_> = (2-r)\bar{\Psi}_1 p + (3-r)\bar{\Psi}_2 p^2 + \cdots .$$

Hence,

$$
\begin{aligned}
D_p\psi_> \cdot \psi_>^{-1} &= D_p\bar\Psi + \frac{1}{2}\{\bar\Psi, D_p\bar\Psi\} + \cdots \\
&= (2-r)\bar\Psi_1 p + \Big((3-r)\bar\Psi_2 + \frac{1}{2}(2-r)(1-r)\bar\Psi_1\bar\Psi_{1,x}\Big)p^2 + \cdots ,
\end{aligned}
$$

From

$$\bar\ell^m = p^m + \bar v_{m,0}p^{m+1} + \bar v_{m,1}p^{m+2} + \cdots$$

we deduce

$$p^m = \bar\ell^m - \bar v_{m,0}\bar\ell^{m+1} - (\bar v_{m,1} - \bar v_{m+1,0}\bar v_{m,0})\bar\ell^{m-2} - \cdots$$

so that

$$\bar m = \cdots + (3-r)\big(\bar\Psi_2 - \frac{1}{2}(2-r)\bar\Psi_1\bar\Psi_{1,x}\big)\bar\ell^2 + (2-r)\bar\Psi_1(x)\bar\ell + x - r\bar t_1\bar\ell^{-1} - (r+1)\bar t_2\bar\ell^{-2} + \cdots .$$

Therefore,

$$\bar M = \mathrm{Ad}_{\psi_{1-r}}\bar m = \cdots - (r+1)\bar t_2\bar L^{-2} - r\bar t_1\bar L^{-1} + X + \bar w_1\bar L + \bar w_2\bar L^2 + \cdots . \tag{60}$$

with

$$
\begin{aligned}
\bar w_1 &= (2-r)\bar\Psi_1(X), \\
\bar w_2 &= (3-r)\big(\bar\Psi_2(X) - \frac{1}{2}(2-r)\bar\Psi_1(X)\bar\Psi_{1,x}(X)\big).
\end{aligned}
$$

□

We now find the Lax equations for M and $\bar M$. For that aim we compute

$$
\begin{aligned}
\partial_n(\psi_< \cdot \exp t)\cdot(\psi_< \cdot \exp t)^{-1} &= \partial\psi_< \cdot \psi_<^{-1} + \mathrm{Ad}_{\psi_<}(p^{n+1-r}) = P_{\geqslant}L^{n+1-r}, \\
\bar\partial_n(\psi_{\geqslant} \cdot \exp \bar t)\cdot(\psi_{\geqslant} \cdot \exp \bar t)^{-1} &= \partial\psi_{\geqslant} \cdot \psi_{\geqslant}^{-1} + \mathrm{Ad}_{\psi_{\geqslant}}(p^{-n+1-r}) = P_<\bar L^{-n+1-r}
\end{aligned}
$$

and conclude that

Proposition 3. *The Lax and Orlov functions are subject to the following Lax equations:*

$$
\begin{cases}
\partial_n L = \{P_{\geqslant}L^{n+1-r}, L\}, & \partial_n\bar L = \{P_{\geqslant}L^{n+1-r}, \bar L\}, \\
\bar\partial_n L = \{P_<\bar L^{1-r-n}, L\}, & \bar\partial_n\bar L = \{P_<\bar L^{1-r-n}, \bar L\},
\end{cases}
$$

$$
\begin{cases}
\partial_n M = \{P_{\geqslant}L^{n+1-r}, M\}, & \partial_n\bar M = \{P_{\geqslant}L^{n+1-r}, \bar M\}, \\
\bar\partial_n M = \{P_<\bar L^{1-r-n}, M\}, & \bar\partial_n\bar M = \{P_<\bar L^{1-r-n}, \bar M\}.
\end{cases}
$$

4.2 Additional symmetries and its generators

Let us consider that the initial conditions $h, \bar h$ in the factorization problem describe smooth curves $h(s) = \exp(H(s)), \bar h(s) = \exp(\bar H(s))$ in G —here $H(s)$ and $\bar H(s)$ are curves in $\mathfrak{g}$—. This imply that the factors $\psi_< = \psi_<(s)$ and $\psi_{\geqslant} = \psi_{\geqslant}(s)$ in the corresponding factorization problemdo depend on s:

$$\psi_<(s)\cdot\exp(t)\cdot h(s) = \psi_{\geqslant}(s)\cdot\exp(\bar t)\cdot\bar h(s). \tag{61}$$

If we introduce the notation

$$\begin{aligned} F &:= \frac{\mathrm{d}h}{\mathrm{d}s}\cdot h^{-1},\\ \bar{F} &:= \frac{\mathrm{d}\bar{h}}{\mathrm{d}s}\cdot \bar{h}^{-1} \end{aligned} \tag{62}$$

and take the right-derivative with respect to s of (61) we get

$$\frac{\mathrm{d}\psi_<}{\mathrm{d}s}\cdot\psi_<^{-1} + \mathrm{Ad}_{\psi_<\cdot\exp t}(F(p,x)) = \frac{\mathrm{d}\psi_\geqslant}{\mathrm{d}s}\cdot\psi_\geqslant^{-1} + \mathrm{Ad}_{\psi_\geqslant\cdot\exp\bar{t}}(\bar{F}(p,x))$$

that implies

$$\frac{\mathrm{d}\psi_\geqslant}{\mathrm{d}s}\cdot\psi_\geqslant^{-1} - \frac{\mathrm{d}\psi_<}{\mathrm{d}s}\cdot\psi_<^{-1} = F(L,M) - \bar{F}(\bar{L},\bar{M}).$$

Now, we may split this equation into

$$\begin{aligned} \frac{\mathrm{d}\psi_<}{\mathrm{d}s}\cdot\psi_<^{-1} &= -P_<(F(L,M) - \bar{F}(\bar{L},\bar{M})),\\ \frac{\mathrm{d}\psi_\geqslant}{\mathrm{d}s}\cdot\psi_\geqslant^{-1} &= P_\geqslant(F(L,M) - \bar{F}(\bar{L},\bar{M})). \end{aligned} \tag{63}$$

Equations (63) imply for the Lax and Orlov functions $L, \bar{L}, M$ and $\bar{M}$ the

Proposition 4. *The Lax and Orlov functions are transformed by the additional symmetries according to the following formulae*

$$\begin{aligned} \frac{\mathrm{d}L}{\mathrm{d}s} &= \{-P_<(F(L,M) - \bar{F}(\bar{L},\bar{M})), L\}, & \frac{\mathrm{d}M}{\mathrm{d}s} &= \{-P_<(F(L,M) - \bar{F}(\bar{L},\bar{M})), M\},\\ \frac{\mathrm{d}\bar{L}}{\mathrm{d}s} &= \{P_\geqslant(F(L,M) - \bar{F}(\bar{L},\bar{M})), \bar{L}\}, & \frac{\mathrm{d}\bar{M}}{\mathrm{d}s} &= \{P_\geqslant(F(L,M) - \bar{F}(\bar{L},\bar{M})), \bar{M}\}. \end{aligned} \tag{64}$$

As we know there is no loss of generality if we take $\bar{h}(s) = \mathrm{id}$, then $\bar{F} = 0$ and (63) read as

$$\frac{\mathrm{d}\psi_<}{\mathrm{d}s}\cdot\psi_<^{-1} = -P_<(F(L,M)), \tag{65}$$

$$\frac{\mathrm{d}\psi_\geqslant}{\mathrm{d}s}\cdot\psi_\geqslant^{-1} = P_\geqslant(F(L,M)). \tag{66}$$

Alternatively, we could set $h(s) = \mathrm{id}$ so that

$$\frac{\mathrm{d}\psi_<}{\mathrm{d}s}\cdot\psi_<^{-1} = P_<(\bar{F}(\bar{L},\bar{M})), \tag{67}$$

$$\frac{\mathrm{d}\psi_\geqslant}{\mathrm{d}s}\cdot\psi_\geqslant^{-1} = -P_\geqslant(\bar{F}(\bar{L},\bar{M})). \tag{68}$$

Hereon we shall assume that

$$F(L,M) = \sum c_{ij}L^iM^j, \quad \bar{F}(\bar{L},\bar{M}) = \sum \bar{c}_{ij}\bar{L}^i\bar{M}^j. \tag{69}$$

Let us keep $t_n = 0$ for $n > N$ and $\bar{t}_n = 0$ for $n > \bar{N}$, then recalling (59) we write

$$M = (N+1-r)t_NL^N + \cdots + (2-r)t_1L + x + w_1L^{-1} + w_2L^{-2} + \cdots, \tag{70}$$

$$\bar{M} = -(r+\bar{N}-1)\bar{t}_{\bar{N}}\bar{L}^{-\bar{N}} - \cdots - r\bar{t}_1\bar{L}^{-1} + X + \bar{w}_1\bar{L} + \bar{w}_2\bar{L}^2 + \cdots \tag{71}$$

Notice that if we want to keep $t_n = 0$ for $n > N$ within the transformation given by the symmetry, then (18) imply that the function $F(L,M)$, when M is expressed as in (70), has no terms proportional to L^n for $n > N - r + 1$. But as F has the form indicated in (69) we only need to impose this condition over each of the products $L^i M^j$

$$\begin{aligned} L^i M^j &= L^i((N+1-r)t_N L^N + \dots + (2-r)t_1 L + x + w_1 L^{-1} + w_2 L^{-2} + \dots)^j \\ &= ((N+1-r)t_N)^j L^{i+Nj} + \dots \Rightarrow c_{ij} = 0 \text{ if } i + Nj > N - r + 1. \end{aligned}$$

Hence,

$$F(L,M) = \sum_{n=1-r}^{N-r+1} \alpha_n\Big(\frac{M}{(N+1-r)L^N}\Big)L^n \tag{72}$$

with α_n analytic functions.

The same reasoning may be applied in order to keep $\bar{t}_n = 0$ for $n > \bar{N}$, and the corresponding condition is

$$\bar{F}(\bar{L},\bar{M}) = \sum_{n=1-r}^{\bar{r}-\bar{N}+1} \bar{\alpha}_n\Big(\frac{\bar{M}}{(\bar{N}-1+r)\bar{L}^{-\bar{N}}}\Big)\bar{L}^{-n} \tag{73}$$

with $\bar{\alpha}_n$ analytic functions.

4.3 Symmetries of the potential r-dmKP equation

In the following lines we shall find three symmetries of the r-dmKP equation (32). Let us suppose that $N = 2$, and $n = 1-r, 2-r$ and $3-r$, so that we have three different contributions, or generators, to F, namely

$$\alpha\big(\frac{M}{(3-r)L^2}\big)L^{1-r},\ \alpha\big(\frac{M}{(3-r)L^2}\big)L^{2-r} \text{ and } \alpha\big(\frac{M}{(3-r)L^2}\big)L^{3-r}.$$

We first observe that

$$\frac{M}{(3-r)L^2} = t_2 + \frac{2-r}{3-r}t_1 L^{-1} + \frac{1}{3-r}xL^{-2} - \frac{r}{3-r}\Psi_1 L^{-3} + \cdots$$

If we denote

$$\varepsilon := \frac{2-r}{3-r}t_1 L^{-1} + \frac{1}{3-r}xL^{-2} - \frac{r}{3-r}\Psi_1 L^{-3} + \cdots$$

we have the following Taylor expansion

$$\begin{aligned} \alpha(t_2+\varepsilon) =& \alpha(t_2) + \dot{\alpha}(t_2)\varepsilon + \frac{1}{2}\ddot{\alpha}(t_2)\varepsilon^2 + \frac{1}{6}\dddot{\alpha}(t_2)\varepsilon^3 + \cdots \\ =& \alpha(t_2) + \frac{2-r}{3-r}\dot{\alpha}(t_2)t_1 L^{-1} + \Big(\frac{1}{3-r}\dot{\alpha}(t_2)x + \frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}(t_2)t_1^2\Big)L^{-2} \\ &+ \Big(-\frac{r}{3-r}\dot{\alpha}(t_2)\Psi_1 + \frac{2-r}{(3-r)^2}\ddot{\alpha}(t_2)t_1 x + \frac{(2-r)^3}{6(3-r)^3}\dddot{\alpha}(t_2)t_1^3\Big)L^{-3} + \cdots. \end{aligned}$$

We shall now analyze each of the three cases

1. Now, we have

$$F = \alpha(t_2+\varepsilon)L^{1-r} = \alpha(t_2)L^{1-r} + \frac{2-r}{3-r}\dot\alpha(t_2)t_1L^{-r} + \cdots .$$

so that

$$\frac{\mathrm{d}\psi_<}{\mathrm{d}s}\cdot\psi_<^{-1} = -P_<F = -\alpha(t_2)P_<(L^{1-r}) - \frac{2-r}{3-r}\dot\alpha(t_2)t_1L^{-r} + \cdots$$

Recalling that

$$\frac{\mathrm{d}\psi_<}{\mathrm{d}s}\cdot\psi_<^{-1} = (\partial_s\Psi_1)p^{-r} + \big(\partial_s\Psi_2 + \frac{r}{2}(\Psi_{1,x}\partial_s\Psi_1 - \Psi_1\partial_s\Psi_{1,x})\big)p^{-r-1} + \cdots$$

we deduce for Ψ_1 the following PDE

$$\Psi_{1,s} = (1-r)\alpha(t_2)\Psi_{1,x} - \frac{2-r}{3-r}\dot\alpha(t_2)t_1$$

whose solution is given by

$$\Psi_1(s) = \frac{2-r}{(1-r)(3-r)}\frac{\dot\alpha(t_2)}{\alpha(t_2)}t_1x + f\Big(t_1,t_2,s+\frac{x}{(1-r)\alpha(t_2)}\Big)$$

with f and arbitrary function.
For $s=0$ we obtain

$$\Psi_1 = \Psi_1(s)\Big|_{s=0} = \frac{2-r}{(1-r)(3-r)}\frac{\dot\alpha(t_2)}{\alpha(t_2)}t_1x + f\Big(t_1,t_2,\frac{x}{(1-r)\alpha(t_2)}\Big)$$

from where we obtain

$$\begin{aligned}\Psi_1(x+(1-r)s\alpha(t_2),t_1,t_2) = \frac{2-r}{(1-r)(3-r)}&\frac{\dot\alpha(t_2)}{\alpha(t_2)}t_1(x+(1-r)s\alpha(t_2))\\ &+ f\Big(t_1,t_2,s+\frac{x}{(1-r)\alpha(t_2)}\Big).\end{aligned}$$

Hence,

$$\Psi_1(s) = -\frac{2-r}{3-r}s\dot\alpha(t_2)t_1 + \Psi_1(x+(1-r)s\alpha(t_2),t_1,t_2).$$

Therefore, we conclude that given any solution $\Psi_1(x,t_1,t_2)$ of the potential r-dmKP equation (32) and any analytic function $\alpha(t_2)$ then

$$\tilde\Psi_1(x,t_1,t_2) := -\frac{2-r}{3-r}\dot\alpha(t_2)t_1 + \Psi_1(x+(1-r)\alpha(t_2),t_1,t_2).$$

is a new solution of the equation. For $u=-\Psi_{1,x}$ the symmetry transformation for a solution of the r-dmKP equation (34) is given by

$$\tilde u(x,t_1,t_2) = u(x+(1-r)\alpha(t_2),t_1,t_2),$$

which for $r=1$ is the identity transformation.

2. In this case

$$\begin{aligned}F =&\alpha(t_2+\varepsilon)L^{2-r}\\ =&\alpha(t_2)L^{2-r}+\frac{2-r}{3-r}\dot\alpha(t_2)t_1L^{1-r}+\Big(\frac{1}{3-r}\dot\alpha(t_2)x+\frac{(2-r)^2}{2(3-r)^2}\ddot\alpha(t_2)t_1^2\Big)L^{-r}+\cdots\end{aligned}$$

so that

$$\begin{aligned}\frac{\mathrm{d}\psi_<}{\mathrm{d}s}\cdot\psi_<^{-1}=\alpha(t_2)\frac{\mathrm{d}\psi_<}{\mathrm{d}t_1}\cdot\psi_<^{-1}-\frac{2-r}{3-r}\dot\alpha(t_2)t_1P_<L^{1-r}&\\ -\Big(\frac{1}{3-r}\dot\alpha(t_2)x+\frac{(2-r)^2}{2(3-r)^2}\ddot\alpha(t_2)t_1^2\Big)L^{-r}+\cdots&.\end{aligned}$$

For Ψ_1 we find the following PDE

$$\Psi_{1,s}=\alpha(t_2)\Psi_{1,t_1}+\frac{(1-r)(2-r)}{3-r}\dot\alpha(t_2)t_1\Psi_{1,x}-\frac{1}{3-r}\dot\alpha(t_2)x-\frac{(2-r)^2}{2(3-r)^2}\ddot\alpha(t_2)t_1^2$$

whose solution is given by

$$\Psi_1(s)=g(x,t_1,t_2)+f\Big(t_2,t_1^2-\frac{2(3-r)}{(1-r)(2-r)}\frac{\alpha}{\dot\alpha}x,s+\frac{t_1}{\alpha}\Big)\tag{74}$$

being f an arbitrary function and

$$g:=\frac{1}{3-r}\frac{\dot\alpha}{\alpha}xt_1+\frac{(2-r)}{6(3-r)^2}\Big((2-r)\frac{\ddot\alpha}{\alpha}-2(1-r)\frac{\dot\alpha^2}{\alpha^2}\Big)t_1^3.$$

Setting $s=0$ in (74) we arrive to

$$\Psi_1=\Psi_1\big|_{s=0}=g(x,t_1,t_2)+f\Big(t_2,t_1^2-\frac{2(3-r)}{(1-r)(2-r)}\frac{\alpha}{\dot\alpha}x,\frac{t_1}{\alpha}\Big).\tag{75}$$

If in (75) we replace the independent variables x and t_1 by

$$\begin{aligned}x(s)&=x+s\Big(\frac{(1-r)(2-r)}{2(3-r)}\dot\alpha(s\alpha+2t_1)\Big),\\ \tilde t_1(s)&=t_1+s\alpha,\end{aligned}$$

we deduce

$$f\big(t_2,t_1^2-\frac{2(3-r)}{(1-r)(2-r)}\frac{\alpha}{\dot\alpha}x,s+\frac{t_1}{\alpha}\big)=\Psi_1(x(s),t_1(s),t_2)-g(x(s),t_1(s),t_2).$$

Hence, from (74) we infer that

$$\Psi_1(s)=g(x,t_1,t_2)-g(x(s),t_1(s),t_2)+\Psi_1(x(s),t_1(s),t_2),$$

and therefore

$$\begin{aligned}\Psi_1(s)=\Psi_1\Big(&x+s\frac{(1-r)(2-r)}{2(3-r)}\dot\alpha(s\alpha+2t_1),t_1+s\alpha,t_2\Big)\\ &-s\frac{1}{3-r}\dot\alpha x-s\frac{(2-r)^2}{2(3-r)^2}\ddot\alpha t_1^2-s^2\frac{2-r}{6(3-r)^2}((1-r)\dot\alpha^2+(2-r)\alpha\ddot\alpha)(3t_1+s\alpha).\end{aligned}\tag{76}$$

Hence, given any solution $\Psi_1(x,t_1,t_2)$ of the potential r-dmKP equation (32) and any analytic function $\alpha(t_2)$ then

$$
\begin{aligned}
\tilde{\Psi}_1 =&\Psi_1\Big(x+\frac{(1-r)(2-r)}{2(3-r)}\dot{\alpha}(\alpha+2t_1),t_1+\alpha,t_2\Big)\\
&-\frac{1}{3-r}\dot{\alpha}x-\frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}t_1^2-\frac{2-r}{6(3-r)^2}((1-r)\dot{\alpha}^2+(2-r)\alpha\ddot{\alpha})(3t_1+\alpha)
\end{aligned}
\tag{77}
$$

is a new solution of (32).
For $u=-\Psi_{1,x}$, thus u solves the r-dmKP equation (34), the corresponding symmetry transformation is given by

$$
\tilde{u}(x,t_1,t_2)=u\Big(x+\frac{(1-r)(2-r)}{2(3-r)}\dot{\alpha}(t_2)(\alpha(t_2)+2t_1),t_1+\alpha(t_2),t_2\Big)+\frac{1}{3-r}\dot{\alpha}(t_2),
$$

which for $r=1$ simplifies to

$$
\tilde{u}(x,t_1,t_2)=u(x,t_1+\alpha(t_2),t_2)+\frac{1}{2}\dot{\alpha}(t_2).
$$

3. Finally, we tackle the most involve case

$$
\begin{aligned}
F =&\alpha(t_2+\varepsilon)L^{3-r}\\
=&\alpha(t_2)L^{3-r}+\frac{2-r}{3-r}\dot{\alpha}(t_2)t_1L^{2-r}+\Big(\frac{1}{3-r}\dot{\alpha}(t_2)x+\frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}(t_2)t_1^2\Big)L^{1-r}\\
&+\Big(-\frac{r}{3-r}\dot{\alpha}(t_2)\Psi_1+\frac{2-r}{(3-r)^2}\ddot{\alpha}(t_2)t_1x+\frac{(2-r)^3}{6(3-r)^3}\dddot{\alpha}(t_2)t_1^3\Big)L^{-r}+\dots
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{\mathrm{d}\psi_<}{\mathrm{d}s}\cdot\psi_<^{-1} =&\alpha(t_2)\frac{\mathrm{d}\psi_<}{\mathrm{d}t_2}\cdot\psi_<^{-1}+\frac{2-r}{3-r}\dot{\alpha}(t_2)t_1\frac{\mathrm{d}\psi_<}{\mathrm{d}t_1}\cdot\psi_<^{-1}\\
&-\Big(\frac{1}{3-r}\dot{\alpha}(t_2)x+\frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}(t_2)t_1^2\Big)P_<L^{1-r}\\
&-\Big(-\frac{r}{3-r}\dot{\alpha}(t_2)\Psi_1+\frac{2-r}{(3-r)^2}\ddot{\alpha}(t_2)t_1x+\frac{(2-r)^3}{6(3-r)^3}\dddot{\alpha}(t_2)t_1^3\Big)L^{-r}+\cdots.
\end{aligned}
\tag{78}
$$

From (78) we deduce that $\Psi_1(s)$ solves the following PDE

$$
\begin{aligned}
\Psi_{1,s} =&\alpha\Psi_{1,t_2}+\frac{2-r}{3-r}\dot{\alpha}t_1\Psi_{1,t_1}+(1-r)\Big(\frac{1}{3-r}\dot{\alpha}x+\frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}t_1^2\Big)\Psi_{1,x}\\
&+\frac{r}{3-r}\dot{\alpha}\Psi_1-\frac{2-r}{(3-r)^2}\ddot{\alpha}t_1x-\frac{(2-r)^3}{6(3-r)^3}\dddot{\alpha}t_1^3.
\end{aligned}
\tag{79}
$$

The general solution of (79) is

$$
\begin{aligned}
\Psi_1(s)=\alpha^{-\frac{r}{3-r}}f\Big(t_1\alpha^{-\frac{2-r}{3-r}},x\alpha^{-\frac{1-r}{3-r}}-\frac{(1-r)(2-r)^2}{2(3-r)^2}t_1^2\alpha^{-2\frac{2-r}{3-r}}\dot{\alpha},s+\int^{t_2}\frac{\mathrm{d}t}{\alpha(t)}\Big)&\\
+g(x,t_1,t_2)&
\end{aligned}
\tag{80}
$$

where f is an arbitrary function and g is given by

$$g(x,t_1,t_2) := \frac{2-r}{(3-r)^2}\frac{\dot\alpha}{\alpha}xt_1 + \frac{(2-r)^3}{6(3-r)^4}\left((3-r)\frac{\ddot\alpha}{\alpha} - (3-2r)\frac{\dot\alpha^2}{\alpha^2}\right)t_1^3.$$

We define

$$c(t) = \int^t \frac{\mathrm{d}t}{\alpha(t)}$$

and define T by the relation

$$c(T) = s + c(t_2),$$

or

$$\int_{t_2}^T \frac{\mathrm{d}t}{\alpha(t)} = s. \tag{81}$$

Observe that from (81) we derive

$$\frac{\mathrm{d}T}{\mathrm{d}t_2}\frac{1}{\alpha(T)} - \frac{1}{\alpha(t_2)} = 0 \Rightarrow \frac{\mathrm{d}T}{\mathrm{d}t_2} = \frac{\alpha(T(t_2))}{\alpha(t_2)} \tag{82}$$

and hence

$$\frac{\mathrm{d}^2T}{\mathrm{d}t_2^2} = \frac{\mathrm{d}T}{\mathrm{d}t_2}\frac{\dot\alpha(T) - \dot\alpha(t_2)}{\alpha(t_2)}. \tag{83}$$

Now, we can write (80) as

$$\Psi_1(s) = \alpha^{-\frac{r}{3-r}} f\Big(t_1\alpha^{-\frac{2-r}{3-r}}, x\alpha^{-\frac{1-r}{3-r}} - \frac{(1-r)(2-r)^2}{2(3-r)^2}t_1^2\alpha^{-2\frac{2-r}{3-r}}\dot\alpha, c(T)\Big) + g(x,t_1,t_2),$$

which setting $s=0$ reads

$$\Psi_1 = \alpha^{-\frac{r}{3-r}} f\Big(t_1\alpha^{-\frac{2-r}{3-r}}, x\alpha^{-\frac{1-r}{3-r}} - \frac{(1-r)(2-r)^2}{2(3-r)^2}t_1^2\alpha^{-2\frac{2-r}{3-r}}\dot\alpha, c(t_2)\Big) + g(x,t_1,t_2).$$

Then, we introduce the following curve

$$\begin{aligned}
t_2(s) :=& T(t_2),\\
t_1(s) :=& \Big(\frac{\alpha(t_2)}{\alpha(T(t_2))}\Big)^{-\frac{2-r}{3-r}} t_1,\\
x(s) :=& \Big(\frac{\alpha(t_2)}{\alpha(T(t_2))}\Big)^{-\frac{1-r}{3-r}} x\\
&+ \frac{(1-r)(2-r)^2}{2(3-r)^2}\alpha(T(t_2))^{\frac{1-r}{3-r}}\alpha(t_2)^{-2\frac{2-r}{3-r}}\big(\dot\alpha(T(t_2)) - \dot\alpha(t_2)\big)t_1^2
\end{aligned}$$

which using (82) and (83) can expressed as

$$\begin{aligned}
t_2(s) :=& T(t_2),\\
t_1(s) :=& (\dot T(t_2))^{\frac{2-r}{3-r}} t_1,\\
x(s) :=& (\dot T(t_2))^{\frac{1-r}{3-r}}\Big(x + \frac{(1-r)(2-r)^2}{2(3-r)^2}\frac{\ddot T(t_2)}{\dot T(t_2)}t_1^2\Big),
\end{aligned} \tag{84}$$

With the curve parameterized as in (84) we see that

$$
\begin{aligned}
f\Big(t_1\alpha^{-\frac{2-r}{3-r}}, x\alpha^{-\frac{1-r}{3-r}}-\frac{(1-r)(2-r)^2}{2(3-r)^2}t_1^2\dot\alpha, s+\int^{t_2}\frac{\mathrm{d}t}{\alpha(t)}\Big)&\\
=\alpha(T)^{\frac{r}{3-r}}(\Psi_1(x(s),t_1(s),t_2(s))-g(x(s),t_1(s),t_2(s))),&
\end{aligned}
$$

and therefore

$$
\Psi_1(s)=g(x,t_1,t_2)-\dot T^{\frac{r}{3-r}}g(x(s),t_1(s),t_2(s))+\dot T^{\frac{r}{3-r}}\Psi_1(x(s),t_1(s),t_2(s)). \tag{85}
$$

Let us evaluate

$$
g(x,t_1,t_2)-\dot T^{\frac{r}{3-r}}g(x(s),t_1(s),t_2(s))=A(t_2)xt_1+B(t_2)t_1^3
$$

where

$$
A:=\frac{2-r}{(3-r)^2}\Big(\frac{\dot\alpha}{\alpha}-\dot T\frac{\dot\alpha(T)}{\alpha(T)}\Big) \tag{86}
$$

$$
\begin{aligned}
B:=&\frac{(2-r)^3}{6(3-r)^4}\Big((3-r)\Big(\frac{\ddot\alpha}{\alpha}-\dot T\frac{\ddot\alpha(T)}{\alpha(T)}\Big)-(3-2r)\Big(\frac{\dot\alpha^2}{\alpha^2}-\dot T^2\frac{\dot\alpha(T)^2}{\alpha(T)^2}\Big)\Big)\\
&-\frac{(2-r)^3}{2(3-r)^4}(1-r)\ddot T\frac{\dot\alpha(T)}{\alpha(T)}.
\end{aligned} \tag{87}
$$

From (82), (83) and (86) we derive

$$
A=-\frac{2-r}{(3-r)^2}\frac{\ddot T}{\dot T}. \tag{88}
$$

A similar expression may be derived, using (82), (83) and its consequences, for B solely in terms of T and its derivatives. However a faster way is to reckon $Axt_1+Bt_1^3$ as a solution of (32) —just by applying the symmetry to $\Psi_1=0$—. In doing so we find that

$$
B=-\frac{(2-r)^2}{6(3-r)}(\dot A+r(3-r)A^2)
$$

and hence, by taking into account (88) we deduce the following expression for B in terms solely of T and its derivatives

$$
B=-\frac{(2-r)^3}{6(3-r)^3}\Big(\frac{\dddot T}{\dot T}-\frac{3}{3-r}\frac{\ddot T^2}{\dot T^2}\Big). \tag{89}
$$

Collecting (85) together with (84), (88) and (89) we deduce the following: If $\Psi(x,t_1,t_2)$ is a solution of the potential r-dmKP equation (32) and $T(t_2)$ is an arbitrary function of t_2 then

$$
\begin{aligned}
\tilde\Psi_1(x,t_1,t_2)=&-\frac{2-r}{(3-r)^2}\frac{\ddot T}{\dot T}xt_1-\frac{(2-r)^3}{6(3-r)^3}\Big(\frac{\dddot T}{\dot T}-\frac{3}{3-r}\frac{\ddot T^2}{\dot T^2}\Big)t_1^3\\
&+\dot T^{\frac{r}{3-r}}\Psi_1\Big(\dot T^{\frac{1-r}{3-r}}\Big(x+\frac{(1-r)(2-r)^2}{2(3-r)^2}\frac{\ddot T}{\dot T}t_1^2\Big),\dot T^{\frac{2-r}{3-r}}t_1,T\Big)
\end{aligned} \tag{90}
$$

is a new solution of (32). As previously for $u=-\Psi_{1,x}$ we have: given a solution u of the r-dmKP equation (34) and an arbitary function $T(t_2)$ then $\tilde{u}$ defined by

$$\tilde{u}=\frac{2-r}{(3-r)^2}\frac{\ddot{T}}{\dot{T}}x+\dot{T}^{\frac{1}{3-r}}u\Big(\dot{T}^{\frac{1-r}{3-r}}\Big(x+\frac{(1-r)(2-r)^2}{2(3-r)^2}\frac{\ddot{T}}{\dot{T}}t_1^2\Big),\dot{T}^{\frac{2-r}{3-r}}t_1,T\Big) \tag{91}$$

is a new solution of (34).

We collect these results regarding the potential r-dmKP equation in the following

Proposition 5. *Given a solution Ψ_1 of the potential r-dmKP equation*

$$\Psi_{1,xt_2}=\frac{3-r}{(2-r)^2}\Psi_{1,t_1t_1}-\frac{(3-r)(1-r)}{2-r}\Psi_{1,xx}\Psi_{1,t_1}-\frac{(3-r)r}{2-r}\Psi_{1,x}\Psi_{1,xt_1}-\frac{(3-r)(1-r)}{2}\Psi_{1,x}^2\Psi_{1,xx}$$

and arbitrary functions $\alpha(t_2)$, $T(t_2)$ the following functions are new solutions of the r-dmKP equation:

$$\begin{aligned}
\tilde{\Psi}_1=&-\frac{2-r}{3-r}\dot{\alpha}(t_2)t_1+\Psi_1(x+(1-r)\alpha(t_2),t_1,t_2),\\
\tilde{\Psi}_1=&-\frac{1}{3-r}\dot{\alpha}(t_2)x-\frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}(t_2)t_1^2-\frac{2-r}{6(3-r)^2}((1-r)\dot{\alpha}(t_2)^2+(2-r)\alpha(t_2)\ddot{\alpha}(t_2))(3t_1+\alpha(t_2))\\
&+\Psi_1\Big(x+\frac{(1-r)(2-r)}{2(3-r)}\dot{\alpha}(t_2)(\alpha(t_2)+2t_1),t_1+\alpha(t_2),t_2\Big)\\
\tilde{\Psi}_1=&-\frac{2-r}{(3-r)^2}\frac{\ddot{T}(t_2)}{\dot{T}(t_2)}xt_1-\frac{(2-r)^3}{6(3-r)^3}\Big(\frac{\dddot{T}(t_2)}{\dot{T}(t_2)}-\frac{3}{3-r}\frac{\ddot{T}(t_2)^2}{\dot{T}(t_2)^2}\Big)t_1^3\\
&+\dot{T}(t_2)^{\frac{r}{3-r}}\Psi_1\Big(\dot{T}(t_2)^{\frac{1-r}{3-r}}\Big(x+\frac{(1-r)(2-r)^2}{2(3-r)^2}\frac{\ddot{T}(t_2)}{\dot{T}(t_2)}t_1^2\Big),\dot{T}(t_2)^{\frac{2-r}{3-r}}t_1,T(t_2)\Big).
\end{aligned}$$

A similar proposition for the r-dmKP equation (34) follows

Proposition 6. *Given a solution u of the r-dmKP equation*

$$u_{t_2}=\frac{3-r}{(2-r)^2}(\partial_x^{-1}u)_{t_1t_1}+\frac{(3-r)(1-r)}{2-r}u_x(\partial_x^{-1}u)_{t_1}+\frac{r(3-r)}{2-r}uu_{t_1}-\frac{(3-r)(1-r)}{2}u^2u_x.$$

and arbitrary functions $\alpha(t_2)$, $T(t_2)$ the following functions are new solutions of the r-dmKP equation:

$$\begin{aligned}
\tilde{u}=&u(x+(1-r)\alpha(t_2),t_1,t_2),\\
\tilde{u}=&\frac{1}{3-r}\dot{\alpha}(t_2)+u\Big(x+\frac{(1-r)(2-r)}{2(3-r)}\dot{\alpha}(t_2)(\alpha(t_2)+2t_1),t_1+\alpha(t_2),t_2\Big)\\
\tilde{u}=&\frac{2-r}{(3-r)^2}\frac{\ddot{T}(t_2)}{\dot{T}(t_2)}t_1+\dot{T}(t_2)^{\frac{1}{3-r}}u\Big(\dot{T}(t_2)^{\frac{1-r}{3-r}}\Big(x+\frac{(1-r)(2-r)^2}{2(3-r)^2}\frac{\ddot{T}(t_2)}{\dot{T}(t_2)}t_1^2\Big),\dot{T}(t_2)^{\frac{2-r}{3-r}}t_1,T(t_2)\Big).
\end{aligned}$$

4.4 Symmetries of the potential r-dDym equation

We shall find three symmetries for the r-dDym equation (44). From (66) we deduce that

$$\frac{\mathrm{d}\bar{\psi}_{1-r}}{\mathrm{d}s}\cdot\bar{\psi}_{1-r}^{-1}+\mathrm{Ad}_{\bar{\psi}_{1-r}}P_{1-r}F(L,M)=0$$

where

$$\frac{\mathrm{d}\bar{\psi}_{1-r}}{\mathrm{d}s}\cdot\bar{\psi}_{1-r}^{-1}=\begin{cases}\dfrac{1}{1-r}\dfrac{\bar{X}_s}{\bar{X}_x}p^{1-r}, & r\neq 1,\\ -\xi_s p^{1-r}, & r=1.\end{cases}$$

As for the previous case we pay particular attention to the case $N=2$, with $n=1-r,2-r$ and $3-r$, so that F can be taken as

$$\alpha\Big(\frac{M}{(3-r)L^2}\Big)L^{1-r},\ \alpha\Big(\frac{M}{(3-r)L^2}\Big)L^{2-r}\text{ and }\alpha\Big(\frac{M}{(3-r)L^2}\Big)L^{3-r},$$

and

$$\begin{aligned}\alpha\Big(\frac{M}{(3-r)L^2}\Big)=&\alpha(t_2)+\frac{2-r}{3-r}\dot{\alpha}(t_2)t_1L^{-1}+\Big(\frac{1}{3-r}\dot{\alpha}(t_2)x+\frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}(t_2)t_1^2\Big)L^{-2}\\&+\Big(-\frac{r}{3-r}\dot{\alpha}(t_2)\Psi_1+\frac{2-r}{(3-r)^2}\ddot{\alpha}(t_2)t_1x+\frac{(2-r)^3}{6(3-r)^3}\dddot{\alpha}(t_2)t_1^3\Big)L^{-3}+\cdots.\end{aligned}$$

1. Let us take $n=1-r$ and

   $$P_{1-r}F(L,M)=\alpha(t_2)p^{1-r},$$

   so that

   $$\mathrm{Ad}_{\bar{\psi}_{1-r}}P_{1-r}F(L,M)=\begin{cases}\dfrac{\alpha(t_2)}{\bar{X}_x}p^{1-r}, & r\neq 1,\\ \alpha(t_2)p^{1-r}, & r=1.\end{cases}$$

   Then, for $r\neq 1$ we have

   $$\bar{X}_s=-(1-r)\alpha(t_2)$$

   and

   $$\bar{X}(s)=\bar{X}-s(1-r)\alpha(t_2).$$

   Hence, given a solution X to the potential r-dDym equation (44) and an arbitrary function $\alpha(t_2)$ the function

   $$\tilde{\bar{X}}=\bar{X}-(1-r)\alpha(t_2),$$

   is a new solution of the potential r-dDym equation.
   When $r=1$ we have

   $$\xi_s=\alpha\Rightarrow\xi(s)=\xi+\alpha(t_2)s.$$

   The corresponding symmetry of (47) is the identity transformation.

2. In this case we set $n = 2 - r$ so that

$$P_{1-r}F(L,M) = \alpha(t_2)P_{1-r}L^{2-r} + \frac{2-r}{3-r}\dot{\alpha}(t_2)t_1p^{1-r}.$$

Recall that

$$\partial_n\bar{\psi}_{1-r} \cdot \bar{\psi}_{1-r}^{-1} + \mathrm{Ad}_{\bar{\psi}_{1-r}} P_{1-r}L^{n+1-r} = 0$$

and therefore

$$\bar{X}_s = \alpha(t_2)\bar{X}_{t_1} - \frac{(1-r)(2-r)}{3-r}\dot{\alpha}(t_2)t_1. \tag{92}$$

The general solution of (92) is

$$\bar{X}(s) = \frac{(1-r)(2-r)}{2(3-r)}\frac{\dot{\alpha}}{\alpha}t_1^2 + f\Big(x, t_2, s + \frac{t_1}{\alpha}\Big) \tag{93}$$

where f is an arbitrary function. Setting $s = 0$ in (93) we obtain

$$\bar{X} = \frac{(1-r)(2-r)}{2(3-r)}\frac{\dot{\alpha}}{\alpha}t_1^2 + f\Big(x, t_2, \frac{t_1}{\alpha}\Big).$$

If

$$t_1(s) = \alpha s + t_1$$

then

$$f\Big(x, t_2, s + \frac{t_1}{\alpha}\Big) = \bar{X}(x, t_1(s), t_2) - \frac{(1-r)(2-r)}{2(3-r)}\frac{\dot{\alpha}}{\alpha}t_1(s)^2$$

so that

$$\bar{X}(s) = -\frac{(1-r)(2-r)}{2(3-r)}\dot{\alpha}(\alpha s + 2t_1)s + \bar{X}(x, t_1 + s\alpha, t_2).$$

Hence, given a solution $\bar{X}$ of the potential r-dDym equation (44) and an arbitrary function $\alpha(t_2)$ the function $\tilde{\bar{X}}$ given by

$$\tilde{\bar{X}} = -\frac{(1-r)(2-r)}{2(3-r)}\dot{\alpha}(\alpha + 2t_1) + \bar{X}(x, t_1 + \alpha, t_2)$$

is a new solution.

When $r = 1$ we obtain

$$\xi_s = \alpha\xi_{t_1} + \frac{\dot{\alpha}}{2}$$

whose solution is

$$\xi(s) = \frac{1}{4}\frac{\dot{\alpha}}{\alpha}t_1^2 + f(x, t_1 + s\alpha, t_2)$$

and the transformation is given by

$$\tilde{\xi} = \frac{1}{4}\dot{\alpha}(\alpha + 2t_1) + \xi(x, t_1 + \alpha, t_2).$$

The corresponding symmetry for the r-dDym equation (47) is

$$\tilde{v} = v(x, t_1 + \alpha(t_2), t_2)$$

3. Now we set $n = 3 - r$ so that

$$
\begin{aligned}
P_{1-r}F(L,M) =& \alpha(t_2)P_{1-r}L^{3-r} + \frac{2-r}{3-r}\dot\alpha(t_2)t_1P_{1-r}L^{2-r} \\
&+ \Big(\frac{1}{3-r}\dot\alpha(t_2)x + \frac{(2-r)^2}{2(3-r)^2}\ddot\alpha(t_2)t_1^2\Big)p^{1-r}
\end{aligned}
$$

and $\bar{X}(s)$ solves

$$
\bar{X}_s = \alpha\bar{X}_{t_2} + \frac{2-r}{3-r}\dot\alpha t_1\bar{X}_{t_1} - \frac{1-r}{3-r}\dot\alpha X - \frac{(1-r)(2-r)^2}{2(3-r)^2}\ddot\alpha t_1^2. \tag{94}
$$

The general solution (94) is

$$
\bar{X}(s) = g(t_1,t_2) + \alpha^{-\frac{1-r}{3-r}} f\Big(x, t_1\alpha^{-\frac{2-r}{3-r}}, s + \int^{t_2}\frac{\mathrm{d}t}{\alpha(t)}\Big)
$$

where f is an arbitrary function and

$$
g(t_1,t_2) := \frac{(1-r)(2-r)^2}{2(3-r)^2}t_1^2\frac{\dot\alpha}{\alpha}.
$$

We now proceed as before; i. e., we define c and T as follows

$$
c(t_2) = \int^{t_2}\frac{\mathrm{d}t}{\alpha(t)}, \quad c(T) = s + c(t_2), \quad \int_{t_2}^{T}\frac{\mathrm{d}t}{\alpha(t)} = s.
$$

For $s = 0$ we have

$$
\bar{X} = g(t_1,t_2) + \alpha^{-\frac{1-r}{3-r}}F(x, t_1\alpha^{-\frac{2-r}{3-r}}, c(t_2))
$$

so that defining the curve

$$
\begin{aligned}
t_1(s) &= \frac{\alpha(T)^{\frac{2-r}{3-r}}}{\alpha(t_2)^{\frac{2-r}{3-r}}}t_1 = \dot{T}^{\frac{2-r}{3-r}}t_1, \\
t_2(s) &= T
\end{aligned}
$$

we have

$$
f(x, t_1\alpha(t_2)^{-\frac{2-r}{3-r}}, c(T)) = \alpha(T)^{-\frac{1-r}{3-r}}\big(X(x,t_1(s),t_2(s)) - g(t_1(s),t_2(s))\big)
$$

and hence

$$
\tilde{X}(s) = g(t_1,t_2) - \dot{T}^{-\frac{1-r}{3-r}}g(t_1(s),t_2(s)) + \dot{T}^{-\frac{1-r}{3-r}}\bar{X}(x,t_1(s),t_2(s))
$$

Let us analyze

$$
\begin{aligned}
g(t_1,t_2) - g(t_1(s),t_2(s)) =& \frac{(1-r)(2-r)^2}{2(3-r)^2}t_1^2\Big(\frac{\dot\alpha(t_2)}{\alpha(t_2)} - \dot{T}\frac{\dot\alpha(T)}{\alpha(T)}\Big) \\
=& -\frac{(1-r)(2-r)^2}{2(3-r)^2}t_1^2\frac{\ddot{T}}{\dot{T}},
\end{aligned}
$$

expression that allows us to write

$$\bar{X}(s) = -\frac{(1-r)(2-r)^2}{2(3-r)^2}t_1^2\frac{\ddot{T}}{\dot{T}} + \dot{T}^{-\frac{1-r}{3-r}}\bar{X}(x, \dot{T}^{\frac{2-r}{3-r}}t_1, T). \tag{95}$$

Thus, given a solution $\bar{X}$ of the potential r-dDym equation (44) and arbitrary function $T(t_2)$ a new solution $\tilde{\bar{X}} = \bar{X}(s)$ is given by (95). The corresponding symmetry of the r-dDym equation (47) is

$$\tilde{v} = \dot{T}^{\frac{1}{3-r}}v(x, \dot{T}(t_2)^{\frac{2-r}{3-r}}t_1, T(t_2))$$

When $r = 1$ we find for ξ the following PDE

$$\xi_s = \alpha\xi_{t_2} + \frac{\dot{\alpha}}{2}t_1\xi_{t_1} + \frac{\dot{\alpha}}{2}x + \frac{\ddot{\alpha}}{8}t_1^2$$

whose solution is

$$\xi(s) = -\frac{x}{2}\log\alpha - \frac{1}{8}\frac{\dot{\alpha}}{\alpha}t_1^2 + F\Big(x, \frac{1}{\sqrt{\alpha}}t_1 + s + \int\frac{\mathrm{d}t}{\alpha(t)}\Big).$$

Finally, from this formula we derive that

$$\tilde{\xi} = \frac{x}{2}\log(\dot{T}(t_2)) + \frac{t_1^2}{8}\frac{\ddot{T}(t_2)}{\dot{T}(t_2)} + \xi\big(x, \sqrt{\dot{T}(t_2)}t_1, T(t_2)\big)$$

is a new solution.

We gather all these results regarding the potential r-dDym equation in the following

Proposition 7. *Given a solution $\bar{X}$ of the potential r-dDym equation ($r \neq 1$)*

$$\bar{X}_{xt_2} = \frac{3-r}{2-r}\Big(\frac{1}{2-r}\bar{X}_{t_1t_1}\bar{X}_x - \frac{1}{1-r}\bar{X}_{xt_1}\bar{X}_{t_1}\Big),$$

and arbitrary functions $\alpha(t_2)$, $T(t_2)$ the following functions are new solutions of the potential r-dDym equation ($r \neq 1$):

$$\begin{aligned}
\tilde{\bar{X}} =& \bar{X}(x, t_1, t_2) - (1-r)\alpha(t_2),\\
\tilde{\bar{X}} =& -\frac{(1-r)(2-r)}{2(3-r)}\dot{\alpha}(t_2)(\alpha(t_2) + 2t_1) + \bar{X}(x, t_1 + \alpha(t_2), t_2),\\
\tilde{\bar{X}} =& -\frac{(1-r)(2-r)^2}{2(3-r)^2}t_1^2\frac{\ddot{T}(t_2)}{\dot{T}(t_2)} + \dot{T}(t_2)^{-\frac{1-r}{3-r}}\bar{X}(x, \dot{T}(t_2)^{\frac{2-r}{3-r}}t_1, T(t_2)).
\end{aligned}$$

Given a solution ξ of the potential $r = 1$ dDym equation

$$\xi_{t_2x} - 2\xi_{t_1t_1} - 2\xi_{t_1}\xi_{t_1x} = 0$$

new solutions $\tilde{\xi}$ are given by

$$\begin{aligned}
\tilde{\xi} =& \alpha(t_2) + \xi(x, t_1, t_2),\\
\tilde{\xi} =& \frac{1}{4}\dot{\alpha}(t_2)(\alpha(t_2) + 2t_1) + \xi(x, t_1 + \alpha(t_2), t_2),\\
\tilde{\xi} =& \frac{x}{2}\log(\dot{T}(t_2)) + \frac{t_1^2}{8}\frac{\ddot{T}(t_2)}{\dot{T}(t_2)} + \xi\big(x, \sqrt{\dot{T}(t_2)}t_1, T(t_2)\big).
\end{aligned}$$

We also resume the results for the r-dDym equation (47)

Proposition 8. *Given a solution v of the r-dDym equation*

$$v_{t_2} = \frac{3-r}{(2-r)^2} v^{r-1}\Big(v^{2-r}\partial_x^{-1}(v^{r-2}v_{t_1})\Big)_{t_1},$$

and arbitrary functions $\alpha(t_2)$, $T(t_2)$ the following functions are new solutions of the r-dDym equation:

$$\begin{aligned}\tilde{v} =& v(x, t_1 + \alpha(t_2), t_2),\\ \tilde{v} =& \dot{T}(t_2)^{\frac{1}{3-r}} v(x, \dot{T}(t_2)^{\frac{2-r}{3-r}} t_1, T(t_2)).\end{aligned}$$

4.5 Symmetries of the r-dToda equation

The r-dToda equation (56a) mixes the independent variables x, t_1 and $\bar{t}_1$. Let us analyze the symmetries associated with the t_1-flow; i. e., study the action of additional symmetries generated by $F(L, M)$. Suppose that $N = 1$, then we have the cases $n = 1-r$ and $2-r$, so that we have two different generators, namely

$$\alpha\big(\frac{M}{(2-r)L}\big)L^{1-r} \text{ and } \alpha\big(\frac{M}{(2-r)L}\big)L^{2-r}.$$

We first observe that

$$\frac{M}{(2-r)L^2} = t_1 + \frac{1}{2-r}xL^{-1} - \frac{r}{2-r}\Psi_1 L^{-2} + \cdots$$

If we denote

$$\varepsilon := \frac{1}{2-r}xL^{-1} - \frac{r}{2-r}\Psi_1 L^{-2} + \cdots$$

we have the following Taylor expansion

$$\begin{aligned}\alpha(t_1+\varepsilon) =& \alpha(t_1) + \dot{\alpha}(t_1)\varepsilon + \frac{1}{2}\ddot{\alpha}(t_1)\varepsilon^2 + \frac{1}{6}\dddot{\alpha}(t_1)\varepsilon^3 + \cdots\\ =& \alpha(t_1) + \frac{1}{2-r}\dot{\alpha}(t_1)xL^{-1} + \Big(-\frac{r}{2-r}\Psi_1\dot{\alpha}(t_1) + \frac{1}{2(2-r)^2}\ddot{\alpha}(t_1)x^2\Big)L^{-2} + \cdots .\end{aligned}$$

Let us study the two cases

1. Now we set

   $$F = \alpha(t_1+\varepsilon)L^{1-r} = \alpha(t_1)L^{1-r} + \frac{1}{2-r}\dot{\alpha}(t_1)xL^{-r} + \cdots .$$

   so that

   $$\bar{X}_s = -(1-r)\alpha(t_1)$$

   and

   $$\bar{X}(s) = \bar{X} - (1-r)s\alpha(t_1).$$

   When $r = 1$ we get

   $$\xi(s) = \xi + s\alpha(t_1)$$

2. In this case we have

$$F=\alpha(t_1+\varepsilon)L^{2-r}=\alpha(t_1)L^{2-r}+\frac{1}{2-r}\dot\alpha(t_1)xL^{1-r}+\cdots.$$

which implies the following PDE for $\bar X$

$$\bar X_s=\alpha(t_1)\bar X_{t_1}-\frac{1-r}{2-r}\dot\alpha(t_1)\bar X$$

whose solution is

$$\bar X(s)=\alpha(t_1)^{\frac{1-r}{2-r}}f\Big(s+\int^{t_1}\frac{\mathrm{d}t}{\alpha(t)}\Big),\qquad \text{with } f \text{ an arbitrary function}$$

which leads to the symmetry

$$\tilde{\bar X}=\dot T^{-\frac{1-r}{2-r}}\bar X(x,T(t_1),\bar t_1).$$

When $r=1$ we get the following PDE for ξ

$$\xi_s=\alpha\xi_{t_1}+\dot\alpha(t_1)x$$

with general solution

$$\xi(s)=-x\log\alpha+f\Big(s+\int^{t_1}\frac{\mathrm{d}t}{\alpha(t)}\Big)$$

leading to the symmetry

$$\tilde\xi=x\log\dot T+\xi(x,T(t_1),\bar t_1).$$

There are additional symmetries associated with the $\bar t_1$ flow. Now $\bar N=1$ and $n=-r$ and $n=1-r$ and there are two different generators:

$$\bar\alpha\big(\frac{\bar M}{-r\bar L^{-1}}\big)\bar L^{1-r}\text{ and }\bar\alpha\big(\frac{\bar M}{-r\bar L^{-1}}\big)\bar L^{-r}.$$

Notice that

$$\frac{\bar M}{-r\bar L^{-1}}=\bar t_1-\bar\varepsilon$$

with

$$\bar\varepsilon:=\frac{1}{r}X\bar L+\frac{2-r}{r}\bar\Psi_1(X)\bar L^2-\cdots,$$

we also have the following Taylor expansion

$$\begin{aligned}\bar\alpha(\bar t_1+\bar\varepsilon)=&\bar\alpha(\bar t_1)-\dot{\bar\alpha}(t_1)\bar\varepsilon+\frac{1}{2}\ddot{\bar\alpha}(t_1)\bar\varepsilon^2+\cdots\\ =&\bar\alpha(\bar t_1)-\frac{1}{r}\dot{\bar\alpha}(\bar t_1)X\bar L+\Big(-\frac{2-r}{r}\bar\Psi_1(X)\dot{\bar\alpha}(\bar t_1)+\frac{1}{2r^2}\ddot{\bar\alpha}(\bar t_1)X^2\Big)\bar L^2+\cdots.\end{aligned}$$

The two generators are

$$\bar F=\bar\alpha(\bar t_1)\bar L^{1-r}, \tag{96}$$

$$\bar F=\bar\alpha(\bar t_1)\bar L^{-r}-\frac{1}{r}\dot{\bar\alpha}(\bar t_1)X\bar L^{1-r}. \tag{97}$$

To deal with the symmetries we recall that

$$\frac{\mathrm{d}\bar{\psi}_{1-r}}{\mathrm{d}s}\cdot\bar{\psi}_{1-r}^{-1}=P_{1-r}\,\mathrm{Ad}_{\bar{\psi}_{1-r}}\,\bar{F}(\bar{L},\bar{M})=P_{1-r}\bar{F}(\bar{\ell},\bar{m}),$$
$$\frac{\mathrm{d}\bar{\psi}_{1-r}}{\mathrm{d}\bar{t}_n}\cdot\bar{\psi}_{1-r}^{-1}=P_{1-r}\,\mathrm{Ad}_{\bar{\psi}_{1-r}}\,\bar{L}^{1-r-n}=P_{1-r}\bar{\ell}^{1-r+n},$$

With the first generator given in (96) we have

$$\frac{1}{1-r}\frac{\bar{X}_s}{\bar{X}_x}=\bar{\alpha}(\bar{t}_1),$$

so that $\bar{X}$ is subject to the following PDE

$$\bar{X}_s=(1-r)\bar{\alpha}(\bar{t}_1)\bar{X}_x$$

whose solution is

$$\bar{X}(s)=f(x+s(1-r)\bar{\alpha}(\bar{t}_1),t_1,\bar{t}_1),$$

with f an arbitrary function. Then, the corresponding symmetry transformation is given by

$$\tilde{\bar{X}}=\bar{X}(x+(1-r)\bar{\alpha}(\bar{t}_1),t_1,\bar{t}_1).$$

For the second generator in (97) we find the following PDE

$$\bar{X}_s=\bar{\alpha}(\bar{t}_1)\bar{X}_{\bar{t}_1}-\frac{1-r}{r}\dot{\bar{\alpha}}(\bar{t}_1)\bar{X}_x$$

whose general solution is

$$\bar{X}(s)=f\Big(x\alpha(\bar{t}_1)^{\frac{1-r}{r}},s+\int^{\bar{t}_1}\frac{\mathrm{d}t}{\alpha(t)}\Big)$$

with f an arbitrary function. Proceeding as before we find a new symmetry:

$$\tilde{\bar{X}}=\bar{X}(\dot{\bar{T}}(\bar{t}_1)^{-\frac{1-r}{r}}x,t_1,\bar{T}(\bar{t}_1))$$

with $\bar{T}$ an arbitrary function on $\bar{t}_1$.

For the $r=1$ case we may proceed as above, however notice that in the $r=1$ case we have the dToda equation, $(\exp(\xi_x))_x+\xi_{t_1\bar{t}_1}=0$, the interchange of t_1 and $\bar{t}_1$ leaves the equation invariant. Thus, the symmetries are as the one derived already for the $F(L,M)$ generators but replacing t_1 by $\bar{t}_1$. Namely:

$$\tilde{\xi}=\xi(x,t_1,\bar{t}_1)+\bar{\alpha}(\bar{t}_1),$$
$$\tilde{\xi}=x\log\dot{\bar{T}}(\bar{t}_1)+\xi(x,t_1,\bar{T}(\bar{t}_1))$$

We collect these results regarding the r-dToda equation in the following

Proposition 9. *Given a solution $\bar{X}$ of the r-dToda equation*

$$\Big((\bar{X}_x)^{-\frac{2-r}{1-r}}\Big)_x-\frac{1}{(1-r)r}\Big(\frac{\bar{X}_{\bar{t}_1}}{\bar{X}_x}\Big)_{t_1}=0,$$

and arbitrary functions $\alpha(t_1)$, $T(t_1)$, $\bar{\alpha}(\bar{t}_1)$ and $\bar{T}(\bar{t}_1)$ the following functions are new solutions of the r-dToda equation:

$$
\begin{aligned}
\tilde{\bar{X}} =&\bar{X}(x,t_1,\bar{t}_1)-(1-r)\alpha(t_1),\\
\tilde{\bar{X}} =&\dot{T}(t_1)^{-\frac{1-r}{2-r}}\bar{X}(x,T(t_1),\bar{t}_1),\\
\tilde{\bar{X}} =&\bar{X}(x+(1-r)\bar{\alpha}(\bar{t}_1),t_1,\bar{t}_1),\\
\tilde{\bar{X}} =&\bar{X}(\dot{\bar{T}}(\bar{t}_1)^{-\frac{1-r}{r}}x,t_1,\bar{T}(\bar{t}_1)).
\end{aligned}
$$

Finally, observe that the symmetries derived for the $r=1$ case, i. e., the Boyer–Finley equation $(\exp(\xi_x))_x+\xi_{t_1\bar{t}_1}=0$, are:

$$
\begin{aligned}
\tilde{\xi} =&\xi(x,t_1,\bar{t}_1)+\alpha(t_1),\\
\tilde{\xi} =&x\log\dot{T}(t_1)+\xi(x,T(t_1),\bar{t}_1),\\
\tilde{\xi} =&\xi(x,t_1,\bar{t}_1)+\bar{\alpha}(\bar{t}_1),\\
\tilde{\xi} =&x\log\dot{\bar{T}}(\bar{t}_1)+\xi(x,t_1,\bar{T}(\bar{t}_1)).
\end{aligned}
$$

These symmetries are the well known, since 1986 by P. Olver, conformal symmetries of the dToda equation.

5 On t_2 invariance and solutions for the potential r-dDym and r-dmKP equations

Here we study the t_2 invariance on the equations potential r-dDym equation (44) and using the Miura type map corresponding solutions for the t_2 invariant solutions of the r-dmKP equation (34), which we may call r dipersionless modified Boussinesq equation.

5.1 t_2-invariance for the r-dDym equation

We analyze here solutions of the (44) which do not depend on one of the variables t_2. Thus, (44) simplifies to

$$(1-r)\bar{X}_{t_1t_1}\bar{X}_x=(2-r)\bar{X}_{xt_1}\bar{X}_{t_1}$$

which can be written as

$$(\log(\bar{X}_{t_1}^{1-r}))_{t_1}=(\log(\bar{X}_x^{2-r}))_{t_1}$$

so that

$$\Big(\log\Big(\frac{\bar{X}_{t_1}^{1-r}}{\bar{X}_x^{2-r}}\Big)\Big)_{t_1}=0.$$

This last equation is equivalent to

$$\frac{1}{k'(x)}\bar{X}_x=\bar{X}_{t_1}^{\frac{1-r}{2-r}}$$

where k' is the derivative of $k(x)$, an arbitrary function of x. Thus, if we introduce the variable $\tilde{x}=k(x)$ we have

$$\bar{X}_{\tilde{x}}=\bar{X}_{t_1}^{\frac{1-r}{2-r}}.$$

This is a first order nonlinear (for $1-r\neq 0$) PDE, and we will solve it by the the method of the complete solution. First observe that a complete integral is

$$\bar{X}(\tilde{x},t_1;a,b)=a^{\frac{1-r}{2-r}}\tilde{x}+at_1+b.$$

Set $b=f(a)$ so that we get a uni-parametric family of solutions depending on arbitrary function f

$$\bar{X}(\tilde{x},t_1;a)=a^{\frac{1-r}{2-r}}\tilde{x}+at_1+f(a).$$

To find the envelope of this solution we request

$$\bar{X}_a=\frac{1-r}{2-r}a^{-\frac{1}{2-r}}\tilde{x}+t_1+f'(a)=0,$$

which determines locally a solution of the form

$$a:=a(\tilde{x},y).$$

Then, the corresponding envelope is given by

$$\bar{X}(x,t_1)=a(k(x),t_1)^{\frac{1-r}{2-r}}k(x)+a(k(x),t_1)t_1+f(a(k(x),t_1))$$

which depends on two arbitrary functions k and f on one variable, being therefore a general solution.

For example, if $f=0$ we get

$$a(\tilde{x},t_1)=\Big(-\frac{1-r}{2-r}\frac{\tilde{x}}{t_1}\Big)^{2-r}$$

and the solution is

$$\bar{X}=\frac{\kappa(x)^{2-r}}{t_1^{1-r}},\qquad \kappa(x):=(-1)^{1-r}\frac{(1-r)^{1-r}}{(2-r)^{2-r}}k(x), \tag{98}$$

Observe that κ may be considered is an arbitrary function.

We may get more general solutions of the potential r-dDym equation (44) which are not t_2 invariant by applying the symmetries given in Proposition 7. We have the following new solutions

$$\begin{aligned}
\tilde{\bar{X}}=&\frac{\kappa(x)^{2-r}}{t_1^{1-r}}-(1-r)\alpha(t_2),\\
\tilde{\bar{X}}=&-\frac{(1-r)(2-r)}{2(3-r)}\dot{\alpha}(t_2)(\alpha(t_2)+2t_1)+\frac{\kappa(x)^{2-r}}{(t_1+\alpha(t_2))^{1-r}},\\
\tilde{\bar{X}}=&-\frac{(1-r)(2-r)^2}{2(3-r)^2}t_1^2\frac{\ddot{T}(t_2)}{\dot{T}(t_2)}+\frac{\kappa(x)^{2-r}}{(\dot{T}(t_2)t_1)^{1-r}}.
\end{aligned}$$

The second and third family are non trivial solutions depending on on two arbitrary functions of one variable each.

The case $f(a)=-\frac{2-r}{r}a^{-\frac{r}{2-r}}$ leads to the equation

$$\alpha^2+\frac{1-r}{2-r}\tilde{x}\alpha+t_1=0,\quad A:=a^{-\frac{1}{2-r}}$$

whose solution is

$$\alpha = -\frac{1-r}{2-r}\frac{\tilde{x}}{2} \pm \sqrt{\frac{(1-r)^2}{(2-r)^2}\frac{\tilde{x}^2}{4} - t_1}$$

and the solution is

$$\bar{X} = \Big(\frac{\tilde{x}}{\alpha} + \frac{t_1}{\alpha^2} - \frac{2-r}{r}\alpha\Big)\alpha^r.$$

Finally, if we consider $f(a) = -\frac{2-r}{1+r}a^{\frac{2-r}{1+r}}$ and introduce the functions

$$F := \sqrt[3]{108t_1 + 12\sqrt{3}\sqrt{4\frac{(1-r)^3}{(2-r)^3}\tilde{x}^3 + t_1^2}}, \quad \alpha := -\frac{1}{6}F + 2\frac{1-r}{2-r}\frac{\tilde{x}}{F}$$

a solution is

$$\bar{X} = \Big(\frac{\tilde{x}}{\alpha} + \frac{t_1}{\alpha^2} - \frac{2-r}{1+r}\alpha\Big)\alpha^r.$$

5.2 Solutions of the r-dmKP through Miura map

As an example we shall use the Miura map

$$-\frac{1}{(1-r)(2-r)}\bar{X}_{t_1}(X(x,t_1,t_2),t_1,t_2) = u(x,t_1,t_2)$$

to get a solution of the r-dmKP equation from the solution of the r-dDym equation as given in (98), which is t_2 independent. The inverse function X is given, in this case in explicit form , by

$$X(x,t_1,t_2) = k^{-1}\bigg(\Big(\frac{xt_1^{1-r}}{C}\Big)^{\frac{1}{2-r}}\bigg), \quad C = (-1)^{1-r}\frac{(1-r)^{1-r}}{(2-r)^{2-r}},$$

and as

$$\bar{X}_{t_1} = -C(1-r)\frac{k(x)^{2-r}}{t_1^{2-r}}$$

we get the corresponding solution of the r-dmKP equation (34)

$$u = \frac{1}{2-r}\frac{x}{t_1}. \tag{99}$$

The corresponding solution of the potential r-dmKP (32) is

$$\Psi_1 = -\frac{1}{2(2-r)}\frac{x^2}{t_1},$$

and applying Proposition 5 we get the following solutions

$$
\begin{aligned}
\tilde{\Psi}_1 =& -\frac{2-r}{3-r}\dot{\alpha}(t_2)t_1 - \frac{1}{2(2-r)}\frac{(x+(1-r)\alpha(t_2))^2}{t_1},\\
\tilde{\Psi}_1 =& -\frac{1}{3-r}\dot{\alpha}(t_2)x - \frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}(t_2)t_1^2\\
&-\frac{2-r}{6(3-r)^2}((1-r)\dot{\alpha}(t_2)^2+(2-r)\alpha(t_2)\ddot{\alpha}(t_2))(3t_1+\alpha(t_2))\\
&-\frac{1}{2(2-r)}\frac{(x+\frac{(1-r)(2-r)}{2(3-r)}\dot{\alpha}(t_2)(\alpha(t_2)+2t_1))^2}{t_1+\alpha(t_2)},\\
\tilde{\Psi}_1 =& -\frac{2-r}{2(3-r)}\frac{\ddot{T}(t_2)}{\dot{T}(t_2)}xt_1 - \frac{(2-r)^3}{2(3-r)^3}\Big(\frac{1}{3}\frac{\dddot{T}(t_2)}{\dot{T}(t_2)} - \frac{1+r}{4}\frac{\ddot{T}(t_2)^2}{\dot{T}(t_2)^2}\Big)t_1^3 - \frac{1}{2(2-r)}\frac{x^2}{t_1}.
\end{aligned}
$$

Observing that

$$\bar{X}_{t_1} = \bar{X}_a a_{t_1} + a$$

and recalling that $\bar{X}_a = 0$ we get the corresponding solution

$$u = -\frac{1}{(1-r)(2-r)}a(X,t_1,t_2)$$

of the potential r-dmKP equation.

6 Twistor equations

Previously we have introduced the Lax and Orlov functions as the following canonical transformations of the pair p, x:

$$
\begin{aligned}
L &= \operatorname{Ad}_{\psi_<\cdot\exp t}p, \quad & \bar{\ell} &= \operatorname{Ad}_{\psi_>\cdot\exp\bar{t}}p, \quad & \bar{L} &= \operatorname{Ad}_{\psi_\geqslant\cdot\exp\bar{t}}p,\\
M &:= \operatorname{Ad}_{\psi_<\cdot\exp t}x, \quad & \bar{m} &:= \operatorname{Ad}_{\psi_>\cdot\exp\bar{t}}x, \quad & \bar{M} &:= \operatorname{Ad}_{\psi_\geqslant\cdot\exp\bar{t}}x.
\end{aligned}
$$

Thus, they satisfy

$$\{L, M\} = L^r, \quad \{\bar{\ell}, \bar{m}\} = \bar{\ell}^r, \quad \{\bar{L}, \bar{M}\} = \bar{L}^r.$$

Another important functions are

$$P := \operatorname{Ad}_h p, \quad Q := \operatorname{Ad}_h x, \quad \bar{P} := \operatorname{Ad}_{\bar{h}} p, \quad \bar{Q} := \operatorname{Ad}_{\bar{h}} x$$

for which we have

$$\{P, Q\} = P^r, \quad \{\bar{P}, \bar{Q}\} = \bar{P}^r.$$

These functions result from the canonical transformation of the p, x variables generated by the initial conditions $h, \bar{h}$ of the factorization problem (4).

We are ready for

Proposition 10. *For any solution $\psi_<$ and $\psi_\geqslant$ of the factorization problem* (4) *the following twistor equations hold*

$$
\begin{aligned}
P(L, M) &= \bar{P}(\bar{L}, \bar{M}),\\
Q(L, M) &= \bar{Q}(\bar{L}, \bar{M}).
\end{aligned}
\tag{100}
$$

Proof. The factorization problem solve

$$\psi_{<} \cdot \exp(t) \cdot h = \psi_{\geqslant} \cdot \exp(\bar{t}) \cdot \bar{h}$$

implies for any function $F(p, x)$

$$\mathrm{Ad}_{\psi_{<}\cdot\exp(t)} \mathrm{Ad}_h F(p, x) = \mathrm{Ad}_{\psi_{\geqslant}\cdot\exp(\bar{t})} \mathrm{Ad}_{\bar{h}} F(p, x).$$

Hence,

$$\mathrm{Ad}_{\psi_{<}\cdot\exp(t)} F(P(p, x), Q(p, x)) = \mathrm{Ad}_{\psi_{\geqslant}\cdot\exp(\bar{t})} F(\bar{P}(p, x), \bar{Q}(p, x))$$

and therefore

$$F(P(L, M), Q(L, M)) = F(\bar{P}(\bar{L}, \bar{M}), \bar{Q}(\bar{L}, \bar{M}))$$

as desired. □

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